International Journal of Electrical and Computer Engineering (IJECE)
Vol. 7, No. 5, October 2017, pp. 2374 – 2381
ISSN: 2088-8708, DOI: 10.11591/ijece.v7i5.pp2374-2381

Accurate Symbolic Steady State Modeling of Buck Converter

Eko Setiawan[1], Takuya Hirata[2], and Ichijo Hodaka[3]
[1]Interdisciplinary Graduate School of Agriculture and Engineering, University of Miyazaki, Japan
[2]Department of Electronic Mechanical Engineering, National Institute of Technology, Oshima College, Japan
[3]Department of Environmental of Robotics, University of Miyazaki, Japan

Article Info

Article history:
Received: Apr 6, 2017
Revised: Jun 9, 2017
Accepted: Jun 23, 2017

Keyword:
Steady state analysis
Fourier series
Switching circuit
Buck converter

ABSTRACT

Steady state analysis is fundamental to any electric and electronic circuit design. Buck converter is one of most popular power electronics circuit and has been analyzed in various situations. Although the behavior of buck converters can be understood approximately by the well-known state space averaging method, little is known in the sense of detailed behavior or exact solution to equations. In this paper a steady state analysis of buck converter is proposed which allows the exact calculation of steady state response. Our exact solution is expressed as a Fourier series. Our result is compared with numerical calculation to be verified. Our method copes with more complicated problems such as describing average power and root-mean-square power that are most critical issues in power electronics circuit.

Corresponding Author:
Ichijo Hodaka
Department of Environmental Robotics, University of Miyazaki
1-1, Gakuen Kibanadai-nishi, Miyazaki, 889-2192, Japan
Phone: +81 985 587 352
E-mail: hijhodaka[at]cc.miyazaki-u.ac.jp

1. INTRODUCTION

An analysis of steady-state response of a system is important key in circuit design and control, included dc-dc converter. In convential method, the steady state of dc-dc converter is assumed as the constant value. Many researches based on state space averaging method and high switching frequency assumption observe steady state response [1, 2, 3, 4, 5]. The method gives simple way to analyze but some ripples are undescribed clearly. The power electronic handbook approximate linear ripple to analyze dc-dc converter more accurately [6]. The approximation may be correct if the switching frequency is high. Since there are some limitation in component, some high frequency is not always reached.

The importance of accurate steady-state analysis has already noticed in many researches [7, 8, 9, 10, 11, 12]. A significant part of the design of circuits requires the simulation of the steady-state response. Parameters such as the gain, harmonic distortion and the input and output impedances are studied in the steady-state mode of operation [7]. Using conventional time-stepping simulations and waiting-time for possible steady state is often not practical because in most cases the time constants of the modes are much larger than the switching period [8]. In conventional method of dc-dc converter analysis, steady state ripple values are negligible, compared to the steady state values themselves. Switching power converters are inherently nonlinear and consequently it is very difficult to calculate the root-mean-square (RMS) values of the state variable ripple. These RMS values are important in order to calculate the current stresses of the different power converter devices as well as to filter design in order to meet the given specifications [9]. Though power electronic handbook [6] shows the RMS calculation using approximation linear ripple, the result is not absolutely correct due to linear approximation. In order to achieve a high performance, proper design and control, it is necessary to have an exact model of converter [10, 11]. High accuracy is one of major features of a good modeling [11].

The dc-dc converter analysis can be classified into two categories, numeric and symbolic analysis. The numeric analysis is found in [7, 8, 10, 12, 13, 14, 15]. The numeric analysis observes the system response by inputting parameter values into model. The numeric analysis needs some computation-time to show the output response. Relation between each parameter is not describe in the equation. The relation between parameter and output is always observed by comparison between parameter change to output change.

The [7] construct impedance or admittance matrix of dc-dc converter. The output response of the converter is calculated by Newton-Raphson. The steady-state output is solved per fixed time-step (fixed sampling interval). The accuracy of analysis is dependent on time-step. Fewer sampling point causes less accuracy. On the other side, more sampling points need more calculation time. The [13] substitute original circuit with periodically switched linear (PSL) circuit. The PSL can be observed by Fourier series. The paper use 110 as sequence number in Fourier series summation. The [14] also use Fourier series to simulate steady-state response. Comparing with [13], the [14] only uses 21 sequences. The [13] applies Fourier series of current switching part. The Fourier series of switching part substitutes original part to be analyzed by Kirchhoff Voltage and Current Law. The paper uses 180 sequence number to draw a steady state response of system. The [12] analyzes buck converter in frequency domain due to high accuracy comparing with conventional time-domain. The paper simulate in three different sequences number that 0, 10 and 100. The paper shows that 10th sequences order is enough to describe steady state response. In Fourier series based method, greater number send the accurate steady-state model but it need more calculation. High order sequence number doesnt bring significant accuracy. Determination of the proper sequence number is another problem beside the main steady-state analysis problem.

Contrary with numerical analysis, symbolic analysis describes relation between parameter and output in the equation. The relation between parameter and output can be observed roughly by the equation. The symbolic analysis is found in [9] and [11]. Symbolic analysis is complicated to be done although it show relation between parameter and output in equation. The [9] perform steady state symbolic analysis to calculate rms value. The paper show solution in a matrix form that is more complicated than ordinary equation. The average and rms calculation include term which is obtained from zero derivative state assumption. Since the ripple is exist and cannot be neglected, this assumption is contrast with early definition. The [11] solve steady-state equation by Laplace transform and revert back into time-domain by inverse of Laplace transform. The solution need the known initial value. The Z-transform is applied due to similarity between initial value and last value in one period. Though the paper show symbolic analysis, the proses is long enough due to calculation of three transformations (Laplace transform, Z-transform and inverse of Laplace transform).

In this paper, an alternative method is proposed that accurately predict and analyze the steady state of switching power converter. The proposed method based on Fourier series since many references show the accuracy [12, 13, 14]. The recovery function is also proposed to generate analysis without dependent in number of sequences orders. The paper is subdivided in several sections to present clearly explanation. Section 2 shows the basic idea of proposed method. Section 3 discusses about implementation of proposed method in buck converter and the recovery function. Complete steady state output function of dc-dc converter is shown in this section. Section 4 verify proposed steady state function by circuit simulation. Finally, the conclusion of this paper is declared in Section 5

2. PROPOSED FOURIER SERIES METHOD

Many phenomena studied in electric circuit are periodic in nature when time goes to infinity. These periodic functions in time *t* can be expressed as Fourier series as in the following equation [16].

$$f(t) = \gamma_0 + \sum_{n\neq 0} \gamma_n e^{jn\omega t} = ... + \gamma_2 e^{j2\omega t} + \gamma_1 e^{j\omega t} + \gamma_0 + \gamma_1 e^{j\omega t} + \gamma_2 e^{j2\omega t} + ... \tag{1}$$

with Fourier coefficients

$$\gamma_0(f) = \frac{1}{T}\int_0^T f(t)dt,\ \gamma_n(f) = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt \tag{2}$$

where T is the period and ω is the angular frequency defined by $\omega = \frac{2\pi}{T}$. In order to help understanding how Fourier series works, let us give an input square-wave function as follows.

$$u(t) = \begin{cases} g & (0 \leq t < Td) \\ 0 & (Td \leq t < T) \end{cases} \quad u(t) = u(t+T) \tag{3}$$

Fourier coefficients of (3) are calculated as follows.

$$\gamma_0(u) = gd,\ \gamma_n(u) = \frac{g}{jn\omega T}\left(1 - e^{jn2\pi d}\right) \tag{4}$$

Then the Fourier series of the equation (3) is written by equation (5).

$$u(t) = gd + \sum_{n\neq 0} \frac{g}{jn2\pi}\left(1 - e^{-jn2\pi d}\right)e^{jn\omega t} \tag{5}$$

An electric system can be described in a simple block diagram as shown in Figure 1. It has been explained that a periodic input has possibility to be analyzed with Fourier series. A periodic input for a transfer function $G(s)$ will generate a periodic output. By knowing its transfer function, the output also can be analyzed into Fourier series. This paper analyzes steady state output in the equation (6). Buck converter can be modelled as a transfer function. Then as we explained the above, buck converter can be expressed as the equation (6).

$$x_{ss}(t) = G(0)\gamma_0(u) + \sum_{n\neq 0} G(jn\omega)\gamma_n(u)e^{jn\omega t} \tag{6}$$

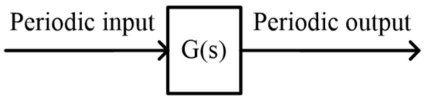


Figure 1. Block diagram

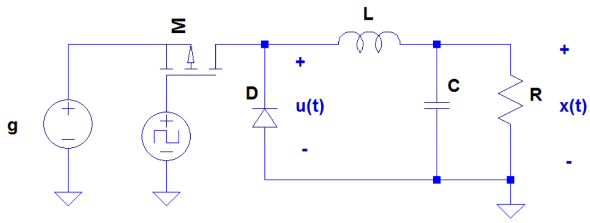
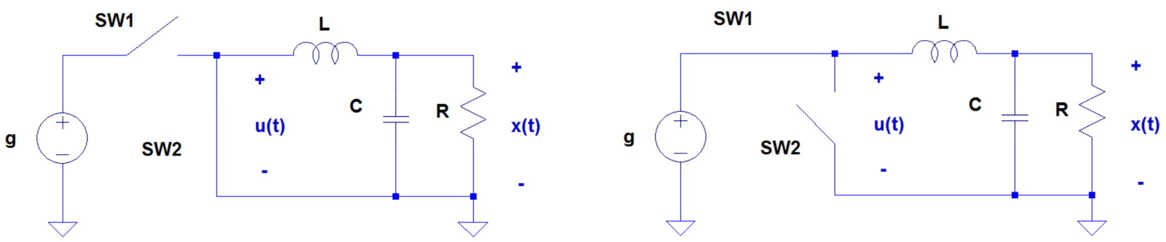


Figure 2. Buck converter circuit

3. BUCK CONVERTER ANALYSIS

Buck converter circuit is shown in Figure 2. Principally, buck converter is driven by two contrary switching ON and OFF. In continuous conduction mode (CCM), buck converter has only two modes. Each mode can be arranged as non-switching circuit by assuming switching part connected (ON) or disconnected (OFF). Figure 3 shows the two modes. Let us assume voltage at switching parts is equal to input function $u(t)$ as shown in Figure 4. Input function $u(t)$ can be described as in the equation (3).



(a) Mode-1 (b) Mode-2

Figure 3. Modes of Buck converter

Based on Figure 4, let $G(s)$ be a transfer function from the input voltage $u(t)$ to the output voltage $x(t)$ as in the following equation.

$$G(s) = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \tag{7}$$

The transfer function (7) has two poles. Here we assume the poles are complex-conjugate pairs ($s = \xi\omega \pm j\eta\omega$), that is,

$$\left(\frac{1}{R^2C^2} - \frac{4}{LC}\right) < 0. \tag{8}$$

With $s = j\omega$, the frequency transfer function can be written as in the following equation.

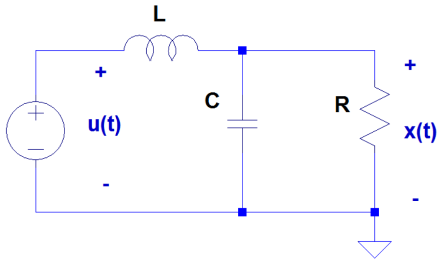


Figure 4. Equivalent circuit

$$G(jn\omega) = \frac{\frac{1}{LC}}{(jn\omega - \xi\omega)^2 + (\eta\omega)^2} \tag{9}$$

Moreover, the transfer function at $s = 0$ is called a DC-gain as follows.

$$G(0) = \frac{\frac{1}{LC}}{0^2 + \frac{1}{RC}0 + \frac{1}{LC}} = 1 \tag{10}$$

In the previous section, Fourier series of square-wave function was explained by the equation (5). Then, Fourier coefficients of the input function (3) were calculated in (4). By assuming that $q = jn$ and substituting (2), (9), and (10) into (6), the proposed equation of buck converter can be expressed as follows.

$$\begin{aligned} x_{ss}(t) &= gd + \sum_{n\neq 0}\left(\frac{\frac{1}{\omega^2 LC}}{(q-\xi)^2+\eta^2}\right)\left(\frac{g}{2\pi q}(1-e^{-2\pi q d})\right)e^{q\omega t} \\ &= gd + \sum_{n\neq 0}\left(\frac{g}{2\pi\omega^2 LC}\right)\left(\frac{1}{(q-\xi)^2+\eta^2}\right)\left(\frac{1}{q}\right)(1-e^{-2\pi q d})e^{q\omega t} \end{aligned} \tag{11}$$

3.1. Recovery function

The infinite series of equation (11) can be represented as follows.

$$\begin{aligned} f_{saw}(t) &= \pi - \omega t, & (0 \le t < T),\ f_{saw}(t) = f_{saw}(t+T) \\ f_c(t) &= \frac{\xi}{\xi^2+\eta^2} + \pi\frac{e^{\xi(\omega t - 2\pi)}\cos(\eta\omega t) - e^{\xi\omega t}\cos(\eta(\omega t - 2\pi))}{\cosh(2\pi\xi) - \cos(2\pi\eta)}, & (0 \le t < T),\ f_c(t) = f_c(t+T) \\ f_s(t) &= \pi\frac{e^{\xi(\omega t - 2\pi)}\sin(\eta\omega t) - e^{\xi\omega t}\sin(\eta(\omega t - 2\pi))}{\cosh(2\pi\xi) - \cos(2\pi\eta)} - \frac{\xi}{\xi^2+\eta^2}, & (0 \le t < T),\ f_s(t) = f_s(t+T) \end{aligned} \tag{12}$$

Fourier coefficients of recovery functions are calculated by (2). By assuming $q = jn$, Fourier series of proposed recovery function is described as follows.

$$\begin{aligned} f_{saw}(t) &= \sum_{n\neq 0}\gamma_n(f_{saw})e^{q\omega t} \\ f_c(t) &= \sum_{n\neq 0}\gamma_n(f_c)e^{q\omega t} \\ f_s(t) &= \sum_{n\neq 0}\gamma_n(f_s)e^{q\omega t} \end{aligned} \tag{13}$$

where

$$\begin{aligned} \gamma_n(f_{saw}) &= \frac{1}{q} & ,\gamma_0(f_{saw}) = 0 \\ \gamma_n(f_c) &= \frac{q-\xi}{(q-\xi)^2+\eta^2} & ,\gamma_0(f_c) = 0 \\ \gamma_n(f_s) &= \frac{\eta}{(q-\xi)^2+\eta^2} & ,\gamma_0(f_s) = 0 \end{aligned} \tag{14}$$

3.2. Time-delay function

Let time-delay function be described as in the following.

$$\tilde{f}(t) = f(t - Th) \tag{15}$$

where T is periodic time and h is delay constant. Fourier coefficients of time-delay function are simply described as

$$\gamma_n(\tilde{f}) = e^{-q2\pi h}\gamma_n(f). \tag{16}$$

3.3. Proposed steady-state function

We can recover function of time expressed by the exponential and trigonometric functions by using our recovery functions (13). The proposed steady state buck converter equation can be also solved by the recovery functions. The summation part of equation (11) can be partially decomposed as follows.

$$\begin{aligned}\left(\frac{1}{(q-\xi)^2+\eta^2}\right)\left(\frac{1}{q}\right) &= \frac{1}{(\eta^2+\xi^2)}\left(\frac{1}{q}+\frac{-q+2\xi}{(q-\xi)^2+\eta^2}\right)\\ &= \frac{1}{\eta^2+\xi^2}\left(\frac{1}{q}-\frac{q-\xi}{(q-\xi)^2+\eta^2}+\left(\frac{\xi}{\eta}\right)\frac{\eta}{(q-\xi)^2+\eta^2}\right)\end{aligned} \tag{17}$$

In the next step, we substitute (17) into (11).

$$x_{ss}(t) = gd+\left(\frac{g}{2\pi\omega^2 LC(\eta^2-\xi^2)}\right)\sum_{n\neq 0}\left[\left(\frac{1}{q}-\frac{q-\xi}{(q-\xi)^2+\eta^2}+\left(\frac{\xi}{\eta}\right)\frac{\eta}{(q-\xi)^2+\eta^2}\right)(1-\mathrm{e}^{-2\pi qd})\mathrm{e}^{q\omega t}\right] \tag{18}$$

Let us assume that

$$m = \frac{g}{2\pi\omega^2 LC(\eta^2-\xi^2)},\ \mu = \frac{\xi}{\eta}. \tag{19}$$

By assuming $q = jn$, the equation (18) may be written as in the following.

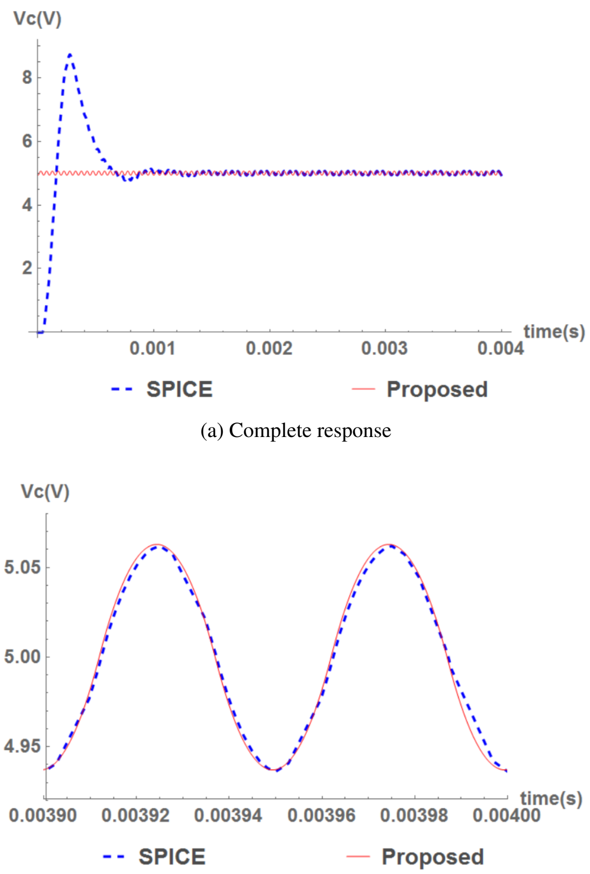
$$\begin{aligned}x_{ss}(t) &= gd + m\sum_{n\neq 0}\frac{1}{q}(1-\mathrm{e}^{-2\pi qd})\mathrm{e}^{q\omega t} - m\sum_{n\neq 0}\frac{q-\xi}{(q-\xi)^2+\eta^2}(1-\mathrm{e}^{-2\pi qd})\mathrm{e}^{q\omega t}\\ &\quad + m\mu\sum_{n\neq 0}\frac{\eta}{(q-\xi)^2+\eta^2}(1-\mathrm{e}^{-2\pi qd})\mathrm{e}^{q\omega t}\\ &= gd + m\left(\underbrace{\sum_{n\neq 0}\gamma_n(f_{saw})\mathrm{e}^{q\omega t}}_{f_{saw}(t)} - \underbrace{\sum_{n\neq 0}\gamma_n(f_{saw})\mathrm{e}^{-2\pi qd}\mathrm{e}^{q\omega t}}_{f_{saw}(t-Td)}\right)\\ &\quad - m\left(\underbrace{\sum_{n\neq 0}\gamma_n(f_c)\mathrm{e}^{q\omega t}}_{f_c(t)} - \underbrace{\sum_{n\neq 0}\gamma_n f_c\mathrm{e}^{-2\pi qd}\mathrm{e}^{q\omega t}}_{f_c(t-Td)}\right)\\ &\quad + m\mu\left(\underbrace{\sum_{n\neq 0}\gamma_n(f_s)\mathrm{e}^{q\omega t}}_{f_s(t)} - \underbrace{\sum_{n\neq 0}\gamma_n(f_s)\mathrm{e}^{-2\pi qd}\mathrm{e}^{q\omega t}}_{f_s(t-Td)}\right)\end{aligned} \tag{20}$$

The equation (20) is easy to be understood if we use the recovery functions as (21).

$$x_{ss}(t) = gd + m\Big(f_{saw}(t) - f_{saw}(t-Td)\Big) - m\Big(f_c(t) - f_c(t-Td)\Big) + m\mu\Big(f_s(t) - f_s(t-Td)\Big) \tag{21}$$

The proposed equation (21) covers infinite series with the recovery functions. Furthermore, the calculation of the average and RMS power are traceable by the recovery functions as follows accurately.

$$\begin{aligned} P_{avg} &= \frac{1}{T}\int_0^T \frac{(x_{ss}(t))^2}{R}dt \\ P_{rms} &= \sqrt{\frac{1}{T}\int_0^T \frac{(x_{ss}(t))^4}{R^2}dt} \end{aligned} \tag{22}$$



(a) Complete response

(b) Steady state response

Figure 5. Comparison between SPICE and proposed analysis of parameter-1

4. SIMULATION RESULT

This section validates our proposed function (21) result by comparing with SPICE (Simulation Program with Integrated Circuit Emphasis). Numerical parameter of dc-dc converter is determined as shown in Table 1.

Numerical parameter of resistance (R), inductance (L) and capacitance (C) give complex-conjugate poles. Numerical calculation utilized mathematical software to plot the proposed steady state response. SPICE simulates the actual circuit responses. Complete response of capacitor voltage by parameter-1 is shown in Figure 5a. Numerical calculation of proposed method is plotted in solid-line while SPICE result in dashed-line. The magnification of steady state response is shown in Figure 5b. Comparison between numerical calculation of proposed function and steady state of SPICE result has similarity in shape and value.

The other complete response of circuit simulation using parameter-2 is shown in Figure 6a and steady state of SPICE and proposed result is described Figure 6b. Figure 5b and 6b describe that numerical calculation of proposed methods give consistent result with SPICE result. The proposed method has advantage in obtaining steady state response without waiting transient time.

Table 1. Numerical parameter

Variable	Parameter-1	Parameter-2
Resistance (R)	6.35 Ω	1.81 Ω
Inductance(L)	100 μH	285 μH
Capacitance (C)	62.7 μF	21.9 μF
Time-period (T)	50 μs	20 μs
Voltage source (g)	10 V	15 V
Duty-ratio (d)	0.5	0.5
Pole (s)	-0.0101 $\pm$ j0.1	-0.0402 $\pm$ j0.0034

5. CONCLUSION

We have shown that the proposed method describes steady-state response directly without calculating transient response. It gives exact calculation and symbolic complete solution of steady state output. Transition between each mode is described clearly by proposed method. Recovery function gives an accurate solution of Fourier series without depending on numerical calculation of summation. Proposed steady state analysis of buck converter has been clarified by comparing with SPICE. Moreover, proposed method makes calculation of power traceable.

REFERENCES

[1] R. D. Middlebrook, S. Ćuk, "A General Unified Approach to Modelling Switching Power Converter Stages," *IEEE PESC Record*, pp. 18-34, 1976.

[2] R. W. Erickson, S. Ćuk, R. D. Middlebrook, "Large Signal Modeling and Analysis of Switching Regulators," *IEEE PESC Record*, pp. 240-250, 1982.

[3] V. Tran, M. MahD, "Modeling and Analysis of Transformerless High Gain Buck-boost DC-DC Converters," *Power Electronics and Drive Systems (IJPEDS), Vol. 4, No. 4*, pp. 528-535, Dec 2014.

[4] J.S. Renius A, V. Kumar K, A. Fredderics, R. Guru, S.L. Nair, "Modelling of Variable Frequency Synchronous Buck Converter," *nternational Journal of Power Electronics and Drive System (IJPEDS) Vol. 5, No. 2*, pp. 237-243, October 2014.

[5] P. Szcześniak, "A Comparison Between Two Average Modelling Techniques of AC-AC Power Converters," *International Journal of Power Electronics and Drive System (IJPEDS), Vol. 6, No. 1*, pp. 32-44, March 2015.

[6] R. W. Erickson, D. Maksimović, *Fundamental of Power Electronics*, Kluwer Academic Publishers, Netherland, 2001.

[7] S. R. Naidu, D. A. Fernandes, "Technique for Simulating the Steady-State Response of Power Electronic Converter," *IET Power Electronics*, Vol. 4, pp. 269-277, 2011.

[8] L. Iannelli, F. Vasca, G. Angelone, "Computation of Steady-State Oscillations in Power Converter Through Complementary," *IEEE Transactions on Circuits and SystemsI: Regular Papers, Vol. 58, No. 6*, pp. 1421-1432. June 2011.

[9] G. T. Kostakis, S. N. Manias, N. I. Margaris, "A Generalized Method for Calculating the RMS Values of Switching Power Converters," *IEEE Transactions on Power Electronics*, Vol. 15 No 4, pp. 616-625, July 2000.

[10] M. Daryaei, M. Ebrahimi, S. A. Khajehoddin, "Accurate Parametric Steady State Analysis and Design Tool for DC-DC Power Converters," *IEEE Applied Power Electronics Conference and Exposition (APEC)*, pp. 2579-2586, 2016.

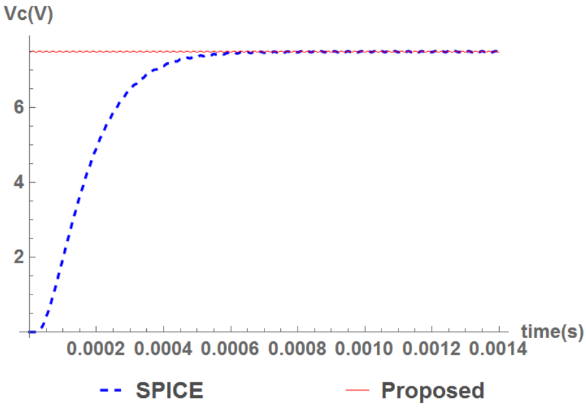
[11] H. M. Mahery, E. Babaei, "Mathematical Modeling of Buck-boost DC-DC Converter and Investigation of Converter Element on Transient and Steady State Responses," *Electrical Power and Energy System 44*, pp. 949-963, 2013.

[12] B. Tsai, O. Trescases, B. Francis, "An Investigation of Steady-State Averaging for the Single-Inductor Dual-Output Buck Converter using Fourier Analysis," *2010 IEEE 12th Workshop on Control and Modeling for Power Electronics (COMPEL)*, June 2010.

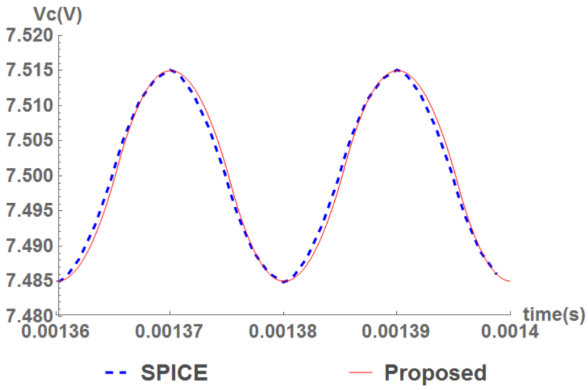
[13] R. Trinchero, I. S. Stievano, F. G. Canavero, "Steady-State Analysis of Switching Power Converter via Augmented Time-Invariant Equivalents," *IEEE Transaction on Power Electronics*, Vol. 29 No 11, pp. 5657-5661, Nov 2014.

[14] F. Mişoc, M. M. Morcos, J. Lookadoo, "Fourier-Series Models of DC-DC Converters," *IEEE 38th North American Power Symposium*, pp. 193-199, 2006.

[15] R. Trinchero, P. Manfredi, I.S.Stievano, F.G. Canavero, "Steady-State Analysis of Switching Converter via



(a) Complete response



(b) Steady state response

Figure 6. Comparison between SPICE and proposed analysis of parameter-2

Frequency-Domain Circuit Equivalents," *IEEE Transactions on Circuits and SystemsII: Express Briefs, Vol. 63, No. 8*, pp. 748-752, August 2016.

[16] G. Strang, *Computational Science and Engineering*, Wellesley-Cambridge Press, USA, 2007.