International Journal of Electrical and Computer Engineering (IJECE)
Vol. 5, No. 2, April 2015, pp. 259~270
ISSN: 2088-8708

Recursive Least-Squares Estimation for the Joint Input-State Estimation of Linear Discrete Time Systems with Unknown Input

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Article Info

Article history:

Received Sep 9, 2014
Revised Jan 3, 2015
Accepted Jan 15, 2015

Keyword:

Information formulas
Inversion lemma
Least-squares
State estimation
Unknown input estimation

ABSTRACT

This paper presents a recursive least-squares approach to estimate simultaneously the state and the unknown input of linear time varying discrete time systems with unknown input. The method is based on the assumption that no prior knowledge about the dynamical evolution of the input is available. The joint input and state estimation are obtained by recursive least-squares formulation by applying the inversion lemmas. The proposed filter is equivalent to recursive three step filter. To illustrate the performance of the proposed filter an example is given.

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1. INTRODUCTION

During the last decades, the problem of unknown input filtering has received growing attention due to its applications in environmental state estimation [1], [2]. The unknown input filtering problem has treated in the literature by different approaches. The first approach assumes that the model for dynamical evolution of the unknown input is available. When the properties of the unknown input are known, the augmented state Kalman filter (ASKF) is a solution. To reduce computation costs of the ASKF, Friedland [2] proposed the two stage Kalman filter where the estimation of the state and unknown input are decoupled. The second approach treats the case when not have a prior knowledge about the dynamical evolution for the unknown input. Kitanidis [1] was the first to solve the problem using the linear unbiased minimum-variance. Darouach et al, [3] extend Kitanidis's filter using a paramaterizing technique to obtain an optimal filter (OEF). Hsieh [4] has developed an equivalent to Kitanidis's filter noted by robust-two stage Kalman filter (RTSKF). Later, Hsieh [5] developed an optimal minimum variance filter (OMVF) to solve the performance of degradation problem encountered in (OEF). Gillijns & De Moor [6] has treated the problem to estimate the state in the presence of unknown input which affects only the systems model. They developed a recursive filter which is optimal in the sense of minimum-variance. This filter has been extended by the same authors [7] for joint input and state estimation to linear discrete-time systems with direct feedthrough where the state and the unknown input estimation are interconnected. This filter is called recursive three step filter (RTSF) and is limited to direct feedthrough matrix has full rank. Cheng et al, [8] proposed a recursive optimal filter with global optimality in the sense of unbiased minimum-variance over all unbiased estimators, but this filter is

limited to estimate the state. Recently, the case of an arbitrary rank has been solved by Hsieh (2009) in the designed optimal filter [9], known as ERTSF (Extend RTSF). Other methods are proposed by Gillijns and Bart de Moor in [10], [11] and [12] which use least -squares (LS) technique and the information formulas.

In this paper, we present an unbiased minimum-variance estimation of the state and the unknown input. These estimates are obtained by solving the information formulas using the weighted least-squares method. The advantage of this method is to provide a direct estimate of the state and unknown input in a single block with a simple calculation.

The paper is organized as follow. Section 2, presents the problem under consideration and some preliminaries. In section 3, we set up the design of the filter equation by recursively solving the weighted least-squares problem. An illustrative example is presented in section 4. Finally, in section 5 we conclude our obtained results.

2. PROBLEM AND PRELEMINARIES

2.1 Problem Formulation

Consider the linear stochastic discrete-time system with unknown input in the following form:

$$x_{k+1} = A_k x_k + G_k d_k + w_k \quad (1)$$

$$y_k = C_k x_k + H_k d_k + v_k \quad (2)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $d_k \in \mathbb{R}^m$ is the unknown input vector and $y_k \in \mathbb{R}^p$ is the measurement vector. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^p$ are assumed to be mutually uncorrelated zeros-mean white random signals with non singular covariance matrices $Q_k = E\left[w_k w_k^T\right] > 0$ and $R_k = E\left[v_k v_k^T\right] > 0$ respectively. The matrices A_k, G_k, C_k and H_k are known and have appropriate dimension. We assume that (A_k, C_k) is observable, $p \geq m$ and the initial state is uncorrelated with the white noises processes w_k and v_k. The initial state x_0 is a Gaussian random variable with $E[x_0] = \hat{x}_0$, $E\left[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T\right] = P_0$ where $E[.]$ denotes the expectation operator. Also, we assume that $rank(C_k G_{k-1}) = rank(G_{k-1})$, the direct feedthrough matrix H_k has an arbitrary *rank* .

The objective of this paper is to design an optimal recursive filter wich estimates both the system state x_k and the unknown input d_k based on the initial estimate $\hat{x}_0$ and the sequence of measurement $\{y_0, y_1, ..., y_k\}$. No prior knowledge about the dynamical evolution of d_k is assumes to be available. Now we derive a Recursive Least Square (RLS) procedure that propagates a one step ahead predicted state estimate. For simplicity of derivations, we use a stochastic approach .We assume that an estimate $\hat{x}_{k/k-1}$ is available with covariance matrix $P_{k/k-1} = E\left[(x_k - \hat{x}_{k/k-1})(x_k - \hat{x}_{k/k-1})^T\right]$ and we seek for a weighted least square (WLS) that allows to estimate $\hat{x}_{k/k}$ based on $\hat{x}_{k/k-1}$ and the newly available measurement y_k .

The error estimation $\tilde{x}_{k/k-1}$ is given by:

$$\tilde{x}_{k/k-1} := x_k - \hat{x}_{k/k-1}. \quad (3)$$

Using (1), (2) and (3), we obtain the following equation:

$$\begin{bmatrix} \hat{x}_{k/k-1} \\ y_k \\ 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ C_k & H_k & 0 \\ A_k & G_k & -I_n \end{bmatrix} \begin{bmatrix} x_k \\ d_k \\ x_{k+1} \end{bmatrix} + \begin{bmatrix} -\tilde{x}_{k/k-1} \\ v_k \\ w_k \end{bmatrix}. \quad (4)$$

So, the corresponding WLS problem is given by

$$\min_{x_k,d_k,x_{k+1}} \left\| \begin{bmatrix} \hat{x}_{k/k-1} \\ y_k \\ 0 \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ C_k & H_k & 0 \\ A_k & G_k & -I_n \end{bmatrix} \begin{bmatrix} x_k \\ d_k \\ x_{k+1} \end{bmatrix} \right\|^2_{\mathcal{W}_k} \tag{5}$$

where $\mathcal{W}_k$ denotes the weighting matrix.

From (5) the interpretation of an MVU (Unbiased Minimum-Variance) estimator is obtained by choosing $\mathcal{W}_k = diag\left(P_{k/k-1}^{-1}, R_k^{-1}, Q_k^{-1}\right)$. The proposed solution of the LS problem (5) is given in the following form:

$$\hat{d}_{k/k} = \mathcal{M}_k \left(y_k - C_k \hat{x}_{k/k-1} \right) \tag{6}$$

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + K_k (y_k - C_k \hat{x}_{k/k-1} - H_k \hat{d}_{k/k}) \tag{7}$$

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k} + G_k \hat{d}_{k/k} \tag{8}$$

Where the gain matrices $\mathcal{M}_k \in \Re^{m\times p}$ and $K_k \in \Re^{n\times p}$ still have to be determined later.

2.2 Preliminaries

The following lemmas are essential for later developments.

Lemma A.1 (The matrix inversion lemma [11]):

Let $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{m\times n}$ and $D \in \Re^{m\times m}$ be real matrices. If A, $D - CA^{-1}B$ and D are non-singular, then $A - BD^{-1}C$ is non-singular, and $\left(A - BD^{-1}C\right)^{-1} = A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)^{-1} CA^{-1}$.

The following formula provides a manner to invert a 2×2 block matrix based on the matrix inversion lemma, $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & 0 \\ 0 & \left(D - CA^{-1}B\right)^{-1} \end{bmatrix} \begin{bmatrix} I & -BD^{-1} \\ -CA^{-1} & I \end{bmatrix}$.

Indeed, the diagonal entries of the first matrix on the right hand side of the equality sign can be computed using the matrix inversion lemma.

Lemma A.2:

Let $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and $C \in \Re^{m\times m}$ be real matrices. If A, C are non-singular then,

$AB\left(C + B^T AB\right)^{-1} = \left(A^{-1} + BC^{-1}B^T\right)^{-1} BC^{-1}$.

3. FILTER DESIGN

The calculation of the optimal matrices $\mathcal{M}_k$ and K_k is addressed in the subsection 3.1 which call the measurement update, yields an estimate of x_k and unknown input d_k. The time update of the state estimation is presented in subsection 3.2.

3.1. Measurement Update

The measurement update is derived from (5) by extracting the rows that depend only on x_k and d_k. This yield,

$$\min_{x_k,d_k} \left\| \begin{bmatrix} \hat{x}_{k/k-1} \\ y_k \end{bmatrix} - \begin{bmatrix} I & 0 \\ C_k & H_k \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} \right\|^2_{\mathcal{W}_{1,k}} \tag{9}$$

Where $\mathcal{W}_{1,k} = diag\left(P_{k/k-1}^{-1}, R_k^{-1}\right)$ denotes the weighting matrix.

Now we derive an explicit update formula by solving the problem state and unknown input estimations. Firstly, note that (9) is equivalent to the least-squares problem

$$\min_{\mathcal{X}_k} \left\| \mathcal{Y}_k - \mathcal{A}_k \mathcal{X}_k \right\|_{\mathcal{W}_{2,k}}^2 \tag{10}$$

Where

$$\mathcal{A}_k = \begin{bmatrix} I & 0 \\ C_k & H_k \end{bmatrix},\ \mathcal{Y}_k = \begin{bmatrix} \hat{x}_{k/k-1} \\ y_k \end{bmatrix},$$
$$\mathcal{X}_k = \begin{bmatrix} x_k \\ d_k \end{bmatrix} \text{ and } \mathcal{W}_{2,k} = diag(R_k^{-1}, P_{k/k-1}^{-1}) \tag{11}$$

Using the Gauss-Markov theorem [13], the solution is written as:

$$\hat{\mathcal{X}}_k = \left(\mathcal{A}_k^T \mathcal{W}_{2,k} \mathcal{A}_k\right)^{-1} \mathcal{A}_k^T \mathcal{W}_{2,k} \mathcal{Y}_k \tag{12}$$

Using (11) the covariance matrix $\left(\mathcal{A}_k^T \mathcal{W}_{2,k} \mathcal{A}_k\right)^{-1}$ follows as

$$\left(\mathcal{A}_k^T \mathcal{W}_{2,k} \mathcal{A}_k\right)^{-1} = \begin{bmatrix} P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k & C_k^T R_k^{-1} H_k \\ H_k^T R_k^{-1} C_k & H_k^T R_k^{-1} H_k \end{bmatrix}^{-1} \tag{13}$$

In the next section we will determinate an unbiased estimate of the state and unknown input by seeking a solution to the equation (13).

Lemma 3.1:

The expression of the error covariance matrix $P_{k/k}^d$ is given by:

$$P_{k/k}^d = \left(H_k^T \tilde{R}_k^{-1} H_k\right)^{-1}, \tag{14}$$

and the error covariance matrix of the state is given in the following form :

$$P_{k/k} = P_{k/k-1} - K_k \left(\tilde{R}_k - H_k P_{k/k}^d H_k^T\right) K_k^T \tag{15}$$

where

$$\tilde{R}_k = C_k P_{k/k-1} C_k^T + R_k \tag{16}$$

$$K_k = P_{k/k-1} C_k^T \tilde{R}_k^{-1} \tag{17}$$

Proof:

Note that, $P_{k/k}$ and $P_{k/k}^d$ can be identified as error covariance matrices of $\hat{x}_{k/k}$ and $\hat{d}_{k/k}$, that is,

$$P_{k/k} = E\left[\tilde{x}_{k/k} \tilde{x}_{k/k}^T\right],$$
$$P_{k/k}^d = E\left[\tilde{d}_{k/k} \tilde{d}_{k/k}^T\right].$$

Where the inverse of $P_{k/k}$ and $P_{k/k}^{d}$ are given, respectively, by

$$P_{k/k}^{-1} = \left(P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k\right) - C_k^T R_k^{-1} H_k (H_k R_k^{-1} H_k)^{-1} \times H_k^T R_k^{-1} C_k, \tag{18}$$

$$\left(P_{k/k}^{d}\right)^{-1} = \left(H_k^T R_k^{-1} H_k\right) - H_k^T R_k^{-1} C_k \times \left(P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k\right)^{-1} C_k^T R_k^{-1} H_k. \tag{19}$$

Then, by applying lemma **A.1** the equation (13) is rewritten as follows:

$$\begin{bmatrix} P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_K & C_k^T R_k^{-1} H_k \\ H_k^T R_k^{-1} C_k & H^T R_k^{-1} H_k \end{bmatrix}^{-1} = \begin{bmatrix} P_{k/k} & 0 \\ 0 & P_{k/k}^{d} \end{bmatrix} \times \begin{bmatrix} I & -C_k^T R_k^{-1} H_k \left(H_k^T R_k^{-1} H_k\right)^{-1} \\ -H_k^T R_k^{-1} C_k \left(P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_K\right)^{-1} & I \end{bmatrix} \tag{20}$$

Applying the matrix inversion lemma **A.1** to the information formulas (18) and (19), the error covariance matrix $P_{k/k}^{d}$ and $P_{k/k}$ are given in the following forms:

$$P_{k/k}^{d} = \left[\left(H_k^T R_k^{-1} H_k\right) - H_k^T R_k^{-1} C_k \left(P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k\right)^{-1} \times C_k^T R_k^{-1} H_k\right]^{-1} \tag{21}$$

$$= \left(H_k^T R_k^{-1} H_k\right)^{-1} + \left(H_k^T R_k^{-1} H_k\right)^{-1} H_k^T R_k^{-1} C_k \times \left[\left(P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k\right) - C_k^T R_k^{-1} H_k \left(H_k^T R_k^{-1} H_k\right)^{-1} H_k^T R_k^{-1} C_k\right]^{-1} \times C_k^T R_k^{-1} C_k \left(H_k^T R_k^{-1} H_k\right)^{-1} \tag{22}$$

$$= H_k^{-1} \left(R_k + C_k P_{k/k-1} C_k^T\right) H_k^{-T} \tag{23}$$

$$= \left(H_k^T \tilde{R}_k^{-1} H_k\right)^{-1} \tag{24}$$

$$P_{k/k} = P_{k/k-1} - K_k \left(\tilde{R}_k - H_k P_{k/k}^{d} H_k^T\right) K_k^T \tag{25}$$

$$P_{k/k} = \left[\left(P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k\right) - C_k^T R_k^{-1} H_k (H_k R_k^{-1} H_k)^{-1} H_k^T R_k^{-1} C_k\right]^{-1} \tag{26}$$

$$= \bar{P}_k + \bar{P}_k C_k^T R_k^{-1} H_k \times \left(H_k^T R_k^{-1} H_k - H_k^T R_k^{-1} C_k \bar{P}_k C_k^T R_k^{-1} H_k\right)^{-1} H_k^T R_k^{-1} C_k \bar{P}_k \tag{27}$$

$$= \bar{P}_k + \bar{P}_k C_k^T R_k^{-1} H_k P_{k/k}^{d} H_k^T R_k^{-1} C_k \bar{P}_k \tag{28}$$

$$= P_{k/k-1} - K_k \tilde{R}_k K_k^T + K_k H_k P_{k/k}^d H_k^T K_k^T \quad (29)$$

where

$$\bar{P}_k = \left(P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k \right)^{-1} \quad (30)$$

$$\bar{P}_k C_k^T R_k^{-1} = P_{k/k-1} C_k^T \tilde{R}_k^{-1} \quad (31)$$

$$C_k P_{k/k-1} = \tilde{R}_k K_k^T \quad (32)$$

The gain matrix K_k that minimize the error covariance is given by

$$K_k = P_{k/k-1} C_k^T \tilde{R}_k^{-1}. \quad (33)$$

Setting the Derivate of (25) with respect to K_k, we get

$$\frac{\partial P_{k/k}}{\partial K_k} = \tilde{R}_k K_k^T + H_k P_{k/k}^d H_k^T K_k^T \quad (34)$$

Let replace the $P_{k/k}^d$ by equation (14), we notice that $\frac{\partial P_{k/k}}{\partial K_k} = 0$, therefore the gain K_k minimize the trace of the matrix covariance $P_{k/k}$.

Lemma 3.2:
An unbiased estimate of the unknown input d_k can be obtained in the following form:

$$\hat{d}_{k/k} = \mathcal{M}_k \left(y_k - C_k \hat{x}_{k/k-1} \right) \quad (35)$$

where

$$\mathcal{M}_k = P_{k/k}^d H_k^T \tilde{R}_k^{-1}. \quad (36)$$

We consider the minimum-variance unbiased state estimation $\hat{x}_{k/k}$ given in the following form:

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + K_k (y_k - C_k \hat{x}_{k-1} - H_k \hat{d}_{k/k}) \quad (37)$$

Proof:
The equation (12) can be written as follows

$$\begin{bmatrix} \hat{x}_{k/k} \\ \hat{d}_{k/k} \end{bmatrix} = \begin{bmatrix} P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_K & C_k^T R_k^{-1} H_k \\ H_k^T R_k^{-1} C_k & H^T R_k^{-1} H_k \end{bmatrix}^{-1} \times \begin{bmatrix} P_{k/k-1}^{-1} & C_k^T R_k^{-1} \\ 0 & H_k^T R_k^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ y_k \end{bmatrix}. \quad (38)$$

Substituting (20) in (38), we obtain

$$\hat{x}_{k/k} = P_{k/k} P_{k/k-1}^{-1} \hat{x}_{k/k-1} \quad (39)$$

$+\left(P_{k/k}C_k^T R_k^{-1} - P_{k/k}C_k^T R_k^{-1}H_k \times (H_k^T R_k^{-1}H_k)^{-1} H_k^T R_k^{-1}\right) y_k$, and estimation of unknown input is given in the following form:

$$\begin{aligned}\hat{d}_{k/k} = P_{k/k}^d H_k^T R_k^{-1} y_k - P_{k/k}^d H_k^T R_k^{-1} C_k (P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k)^{-1} \\ \times (P_{/k-1}^{-1} x_{k/k-1} + C_k^T R_k^{-1} y_k).\end{aligned} \tag{40}$$

Let us apply the lemma **A.1** and **A.3** to the equation (39) and (40), the estimate of the unknown input is given by:

$$\begin{aligned}\hat{d}_{k/k} = P_{k/k}^d H_k^T R_k^{-1} y_k - P_{k/k}^d H_k^T R_k^{-1} C_k \\ \times (P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k)^{-1} (P_{/k-1}^{-1} \hat{x}_{k/k-1} + C_k^T R_k^{-1} y_k).\end{aligned} \tag{41}$$

where

$$\begin{aligned}P_{k/k}^d H_k^T R_k^{-1} C_k (P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k)^{-1} P_{k/k-1}^{-1} \hat{x}_{k/k-1} \\ = \left(H_k^T \tilde{R}_k^{-1} H_k\right) H_k^T \tilde{R}_k^{-1} C_k \hat{x}_{k/k-1}.\end{aligned} \tag{42}$$

$$\begin{aligned}P_{k/k}^d H_k^T \left[R_k^{-1} - R_k^{-1} C_k \left(P_{k/k-1}^{-1} + C_k^T R_k^{-1} C_k\right)^{-1} C_k^T R_k^{-1}\right] y_k \\ = P_{k/k}^d H_k^T \left(R_k + C_k P_{k/k-1} C_k^T\right)^{-1} y_k.\end{aligned} \tag{43}$$

$$\hat{d}_{k/k} = \left(H_k^T \tilde{R}_k^{-1} H_k\right) H_k^T \tilde{R}_k^{-1} \left(y_k - C_k \hat{x}_{k/k-1}\right) \tag{44}$$

Remark 3.1: to evaluate the performance of the filter in case where H_k has an arbitrary rank we use the heuristic extension presented in [9] by replacing equation (14) and (36) by:

$$\mathcal{M}_k = P_{k/k}^d H_k^T \tilde{R}_k^{-1}, \quad P_{k/k}^d = \left(H_k^T \tilde{R}_k^{-1} H_k\right)\dagger \tag{45}$$

where † the Moor-Penrose is generalized inverse

$$M^\dagger = \left(M^T M\right)^{-1} M^T$$

The state estimation $\hat{x}_{k/k}$ is given in the following form:

$$\begin{aligned}\hat{x}_{k/k} = P_{k/k} P_{k/k-1}^{-1} \hat{x}_{k/k-1} + \\ \left(P_{k/k}C_k^T R_k^{-1} - P_{k/k}C_k^T R_k^{-1} H_k (H_k^T R_k^{-1} H_k)^{-1} H_k^T R_k^{-1}\right) y_k\end{aligned} \tag{46}$$

Using the inversion lemma **A.1** and **A.3** we can show that:

$$\begin{aligned}P_{k/k} P_{k/k-1}^{-1} \hat{x}_{k/k-1} = \hat{x}_{k/k-1} - P_{k/k-1} C_k^T \tilde{R}_k^{-1} C_k \hat{x}_{k/k-1} \\ + P_{k/k-1} C_k^T \tilde{R}_k^{-1} H_k (H_k^T \tilde{R}_k^{-1} H_k)^{-1} H_k^T \tilde{R}_k^{-1} C_k \hat{x}_{k/k-1}.\end{aligned} \tag{47}$$

$$\begin{aligned}P_{k/k} C_k^T \left(R_k^{-1} - R_k^{-1} H_k (H_k^T R_k^{-1} H_k)^{-1} H_k^T R_k^{-1}\right) y_k \\ = P_{k/k-1} C_k^T \tilde{R}_k^{-1} y_k - P_{k/k-1} C_k^T \tilde{R}_k^{-1} H_k (H_k^T \tilde{R}_k^{-1} H_k)^{-1} \\ \times H_k^T \tilde{R}_k^{-1} y_k.\end{aligned} \tag{48}$$

$$\begin{aligned}\hat{x}_{k/k} &= \hat{x}_{k/k-1} - P_{k/k-1}C_k^T\tilde{R}_k^{-1}C_k\hat{x}_{k/k-1} \\ &+ P_{k/k-1}C_k^T\tilde{R}_k^{-1}H_k(H_k^T\tilde{R}_k^{-1}H_k)^{-1}H_k^T\tilde{R}_k^{-1}C_k\hat{x}_{k/k-1} \\ &+ P_{k/k-1}C_k^T\tilde{R}_k^{-1}y_k - P_{k/k-1}C_k^T\tilde{R}_k^{-1}H_k(H_k^T\tilde{R}_k^{-1}H_k)^{-1} \\ &\times H_k^T\tilde{R}_k^{-1}y_k.\end{aligned} \tag{49}$$

$$\begin{aligned}\hat{x}_{k/k} &= \left(I - P_{k/k-1}C_k^T\tilde{R}_k^{-1}C_k\right)\hat{x}_{k/k-1} + P_{k/k-1}C_k^T\tilde{R}_k^{-1}y_k \\ &-P_{k/k-1}C_k^T\tilde{R}_k^{-1}H_k(H_k^T\tilde{R}_k^{-1}H_k)^{-1}H_k^T\tilde{R}_k^{-1}\left(y_k - C_k\hat{x}_{k/k-1}\right)\end{aligned} \tag{50}$$

3.2. Time Update

Firstly, we extract from (4) the equation that depends on x_{k+1}.

$$x_{k+1} = A_k x_k + G_k d_k + w_k \tag{51}$$

Second substituting x_k and d_k for their LS estimates $\hat{x}_{k/k}$ and $\hat{d}_{k/k}$ obtained during the measurement update (41) and (50). Then, we obtain

$$A_k\hat{x}_{k/k} + G_k\hat{d}_{k/k} = x_{k+1} - (A_k\tilde{x}_{k/k} + G\tilde{d}_{k/k} + w_k) \tag{52}$$

The corresponding LS problem is given by

$$\min\left\|x_{k+1} - A_k\hat{x}_{k/k} - G_k\hat{d}_k\right\|_{\mathcal{W}_{3,k}} \tag{53}$$

Where $\mathcal{W}_{3,k}$ denotes the weighting matrix which we choose

$$\mathcal{W}_{3,k} = \left(E\left[\left(A_k\tilde{x}_{k/k} + G_k\tilde{d}_{k/k} + w_k\right)\left(A_k\tilde{x}_{k/k} + G_k\tilde{d}_{k/k} + w_k\right)^T\right]\right)^{-1} \tag{54}$$

From equation (53)

$$\hat{x}_{k+1/k} = A_k\hat{x}_{k/k} + G_k\hat{d}_{k/k} \tag{55}$$

The error estimation $\tilde{x}_{k+1/k}$ is given by

$$\tilde{x}_{k+1/k} = x_{k+1} - \hat{x}_{k+1/k} \tag{56}$$

$$= A_k\tilde{x}_{k/k} + G_k\tilde{d}_{k/k} + w_k \tag{57}$$

In consequence, the covariance matrix of $\hat{x}_{k+1/k}$ is given by:

$$P_{k+1/k} = E\left[\tilde{x}_{k+1/k}\ x_{k+1/k}^T\right] \tag{58}$$

$$= \begin{bmatrix}A_k & G_k\end{bmatrix}\begin{bmatrix}P_{k/k} & P_{k/k}^{xd} \\ P_{k/k}^{dx} & P_{k/k}^{d}\end{bmatrix}\begin{bmatrix}A_k^T \\ G_k^T\end{bmatrix} + Q_k \tag{59}$$

It follows from (35) that $\tilde{d}_{k/k}$ is given by:

$$\tilde{d}_{k/k} = \left(I - M_k H_k\right) d_k - M_k \left(C_k \tilde{x}_{k/k-1} + v_k\right) \quad (60)$$
$$P_{k/k}^{xd} = \left(P_{k/k}^{xd}\right)^T = E\left[\tilde{x}_{k/k} \tilde{d}_{k/k}\right].$$

Using (56) and (60), it follows that

$$P_{k/k}^{xd} = -K_k H_k P_{k/k}^{d} \quad (61)$$

4. ILLUSTRATIVE EXAMPLE

To show the proposed results, the numerical example given by Darouach, Zasadzinki and Boutayeb (2003) is considered, where the parameters of systems (1) and (2) are given as follows, the parameters of the system (1) and (2) are given by:

$$A_k = \begin{bmatrix} -0.0005 & -0.0084 \\ 0.0517 & 0.8069 \end{bmatrix}, C_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$G_k = \begin{bmatrix} 0.0129 & 0 \\ -1.2504 & 0 \end{bmatrix}, Q_k = \begin{bmatrix} 0.0036 & 0.0342 \\ 0.0342 & 0.3249 \end{bmatrix} R_k = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.16 \end{bmatrix}.$$

Without loss of generality, the initial state and its estimate are both assumed to be zero, and the initial covariance is given by $P_0 = diag(10, 200)$. The unknown input are given by

$$d_k = \begin{bmatrix} 5u_s[k] - 5u_s[k-20] + 5u_s[k-70] \\ 4u_s[k] - 4u_s[k-30] + 4u_s[k-65] \end{bmatrix}$$

where $u_s[k]$ is the unit-step function. In this example, we assume that the simulation time is 100 time step.

$$H_k^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, H_k^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } H_k^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Case 1: $H_k = H_k^1$

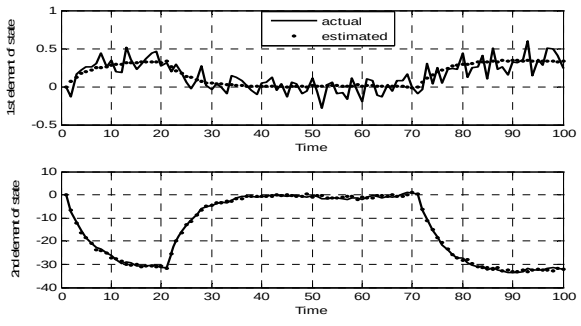

Figure 1. Actual and estimated value of the state

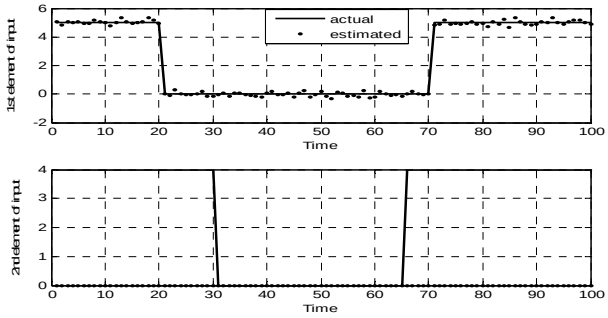

Figure 2. Actual and estimated value of the input

Case 2: $H_k = H_k^2$

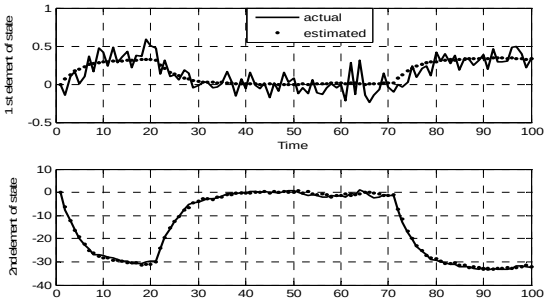

Figure 3. Actual and estimated value of the state

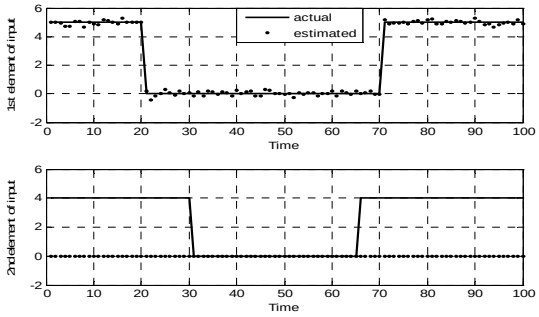

Figure 4. Actual and estimated value of the input

Case 3: $H_k = H_k^3$

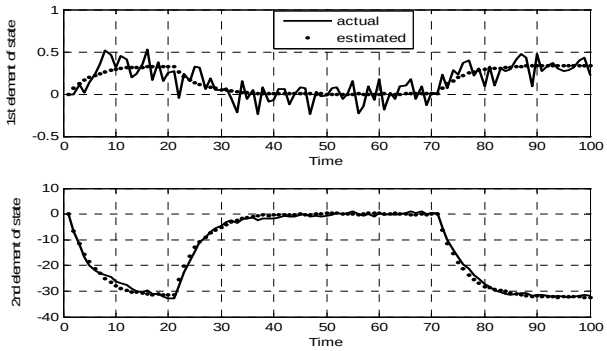

Figure 5. Actual and estimated value of the state

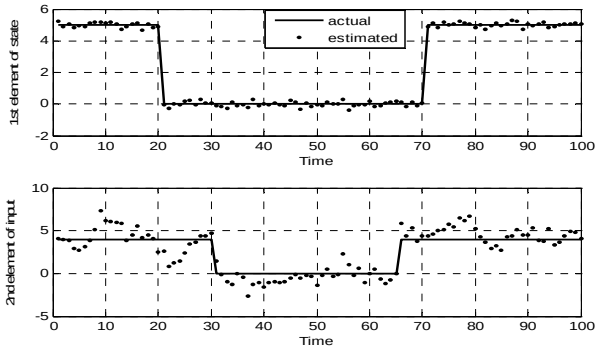

Figure 6. Actual and estimated value of the input

Table 1. Performance of the proposed filter

H_k	*RMSE*				*Trace P*	
	$\tilde{x}_k^1$	$\tilde{x}_k^2$	$\tilde{d}_k^1$	$\tilde{d}_k^2$	$tracP_k^x$	$tracP_k^d$
H_k^1	0.031	0.161	0.024	-	0.430	0.0135
H_k^2	0.083	0.064	0.120	4.000	0.459	0.0136
H_k^3	0.071	0.014	0.018	0.123	1.014	1.1847

In table 1, the root square errors (RMSE) of the state $x_k = \begin{bmatrix} x_k^1 & x_k^2 \end{bmatrix}^T$ and the inputs $d_k = \begin{bmatrix} d_k^1 & d_k^2 \end{bmatrix}^T$ are given as well as the traces of their steady-input and state estimation error covariance. For example the RMSE of the first component of sate vector is calculated by

$$RMSE\left(\tilde{x}_{1,k}\right) = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(x_{1,k} - \hat{x}_{1,k}\right)^2}$$

In figures 1, 3 and 5 we plot that actual and the estimated value of the two element of the state vector $x_k = \begin{bmatrix} x_k^1 & x_k^2 \end{bmatrix}^T$ in the three cases.

In figures 2, 4 and 6 we plot that actual and the estimated value of the two elements of the first and second element of the input vector $d_k = \begin{bmatrix} d_k^1 & d_k^2 \end{bmatrix}^T$ in the three cases.

According to Table1, Figures 1, 2, 3, 4, 5 and 6 we may conclude the following results:
If the direct feedthrough matrix has full rank $H_k=H_k^3$ then the proposed filter (RLSF) guarantees an estimate with a small value (RMSE) for the two components of the two vectors of state and unknown input.

5. CONCLUSION

In this paper, the recursive filter design for systems with unknown input via the least-squares technique is proposed. The obtained recursive filter is named RLSF. This solution is based on the technique least square when the direct feedthrough matrix of the unknown input has an arbitrary rank. An application of the proposed filter has been shown by an illustrative example. This filter may be used in resolving Fault Detection and Isolation.

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