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New approximations for the numerical radius of an $n \times n$ operator matrix

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ABSTRACT

Many mathematicians have been interested in establishing more stringent bounds on the numerical radius of operators on a Hilbert space. Studying the numerical radii of operator matrices has provided valuable insights using operator matrices. In this paper, we present new, sharper bounds for the numerical radius $\frac{1}{4}\left\||A|^2+|A^*|^2\right\| \leq w^2(A) \leq \frac{1}{2}\left\||A|^2+|A^*|^2\right\|$, that found by Kittaneh. Specifically, we develop a new bound for the numerical radius $w(T)$ of block operators. Moreover, we show that these bounds not only improve upon but also generalize some of the current lower and upper bounds. The concept of finding and understanding these bounds in matrices and linear operators is revisited throughout this research. Furthermore, the study emphasizes the importance of these bounds in mathematics and their potential applications in various mathematical fields.

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1. INTRODUCTION

Consider a complex Hilbert space $\mathcal{H}$. The C^*-algebra of all bounded linear operators on $\mathcal{H}$ is denoted as $\mathcal{B}(\mathcal{H})$. Throughout this paper, we define the real part $\mathrm{Re}(.)$, the imaginary part $\mathrm{Im}(.)$, the absolute value $|.|$, the numerical radius $w(.)$, and the standard operator norm $||.||$ for $A \in \mathcal{B}(\mathcal{H})$ as follows:

$$\mathrm{Re}(A) = \frac{A+A^*}{2}, \quad \mathrm{Im}(A) = \frac{A-A^*}{2i}, \quad |A| = (A^*A)^{\frac{1}{2}},$$

$$w(A) = \sup\left\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\right\}, \quad \|A\| = \sup\left\{\sqrt{\langle Ax, Ax\rangle} : x \in \mathcal{H}, \|x\| = 1\right\}.$$

It is known that the norms $w(.)$ and $||.||$ are equivalent and satisfy

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|, \tag{1}$$

for every $A \in \mathcal{B}(\mathcal{H})$. With a few exceptions, it is typically challenging to determine the precise value of the numerical radius $w(A)$ for any matrix $A \in \mathcal{B}(\mathcal{H})$. As a result, researchers have worked on determining tighter upper and lower bounds of $w(A)$ for general matrices A, better than those presented in (1). Readers can consult [1]–[5] and the references therein for the latest developments on the upper and lower bounds of the numerical radius. The authors investigated numerical radius inequalities for sectorial matrices, a particular type of matrix, in [6]–[8]. They created several more precise upper and lower bounds for the numerical radius of these matrices. The study in [9] explains the basics of Schur complements in the context of a particular numerical radius problem. New inequalities for a generalized numerical radius of block operators are presented in [10]. An improved bound for the numerical radius of $n \times n$ operator matrices is developed in [11]. Additionally, they give numerical radius bounds for the product of two operators and the commutator of operators. New upper and lower bounds for the numerical radii of some operator matrices are found by the authors in [12]. In [13], a new bound for polynomial zeros is derived. Understanding norm-related and numerical radius-related inequalities is crucial when performing mathematical analysis. This knowledge offers important insights into operator behavior and approximation accuracy. A comprehensive review of several inequalities concerning the Euclidean operator radius is given in [14]. It covers addition and multiplication for groups of n-tuple operators. The need for more stringent and general bounds on the numerical radius of operators on Hilbert spaces is discussed in this paper. The effectiveness of existing upper and lower bounds in a variety of applications is limited because they are frequently sub-optimal, especially for block operator matrices. Our findings provide a more thorough understanding of numerical radius behavior, particularly for block operators, by both improving and generalizing current estimates. The enhanced bounds complement theoretical advancements in functional analysis and operator theory. With possible uses in mathematical physics, quantum mechanics, and numerical analysis, they offer improved analytical tools for researching operator matrices. Our method ensures wider applicability by methodically generalizing existing upper and lower bounds, in contrast to earlier studies that only provide isolated bounds. Advanced inequalities arguments are used to construct theoretical proofs, guaranteeing the generality and robustness of our findings. In particular, we improve estimates for block operators by deriving new, sharper bounds for the numerical radius of operators. These bounds offer a more comprehensive framework for numerical radius analysis by both improving and generalizing previous findings. The enhanced bounds may result in more accurate numerical techniques for linear algebra and functional analysis, which would be advantageous for computational mathematics. Applications for the findings could be found in control theory stability analysis, quantum mechanics, and other domains where operator. This work opens the door for further exploration of numerical radius bounds in more complex operator classes, potentially inspiring future research on non-normal operators and unbounded operators in Hilbert spaces. Numerous mathematicians have aimed to improve the inequalities in (1). For example, see [15]–[20]. In [16], Kittaneh refined the inequality in (1) by proving that if $A \in \mathcal{B}(\mathcal{H})$, then

$$\frac{1}{4}\left\||A|^2+|A^*|^2\right\| \le w^2(A) \le \frac{1}{2}\left\||A|^2+|A^*|^2\right\|. \tag{2}$$

Let $\mathcal{H}_1, \mathcal{H}_2 ...., \mathcal{H}_n$ be complex Hilbert spaces, and let $\mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ be the space of all bounded linear operators from $\mathcal{H}_j$ into $\mathcal{H}_i$. Based on this structure, any operator $T \in \mathcal{B}\left(\overset{n}{\underset{i=1}{\oplus}} \mathcal{H}_i\right)$ (where $\overset{n}{\underset{i=1}{\oplus}} \mathcal{H}_i$ is the direct sum of $\mathcal{H}_i, i = 1, 2, ..., n$) can be represented by an $n \times n$ operator matrix $T = [T_{ij}]$, where $T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$. To discover more important results related to the numerical radius of operator matrices, see [21]–[26]. The results given in [15] motivated us to develop new lower and upper bounds for $n \times n$ operator matrices. In particular, we show that if $T = \begin{bmatrix} 0 & & \Lambda_1 \\ & \Lambda_2 & \\ \iddots & & \\ \Lambda_n & & 0 \end{bmatrix}$, where $\{\Lambda_i\}_{i=1}^n \subseteq \mathcal{B}(\mathcal{H})$, and $\Lambda_1, \Lambda_2, ..., \Lambda_n \in \mathcal{H}$, then

$$w(T) \ge \frac{1}{4}\max_{1\le i\le n}\left\||\Lambda_i|^2+|\Lambda_{n-i+1}^*|^2\right\| + \frac{1}{8}\max_{1\le i\le n}\left|||\Lambda_i+\Lambda_{n-i+1}^*|| - ||\Lambda_i-\Lambda_{n-i+1}^*||\right|$$

and

$$w(T) \le \frac{1}{4}\max_{1\le i\le n}\left\||\Lambda_i^*|^2+|\Lambda_{n-i+1}|^2\right\| + \frac{1}{2}\max_{1\le i\le n} w\left(|\Lambda_{n-i+1}^*|\,|\Lambda_i|\right).$$

As special cases of these bounds, we will refine the inequalities in (1) and (2). We will also provide some concrete examples showing how these new bounds improve upon those in (1) and (2).

2. BACKGROUND PRELIMINARIES

In this section, some key results about the numerical radius and the operator norm on a complex Hilbert space are reviewed. These results are essential for proving our main findings. The following lemma describes the numerical radius of an operator in terms of the numerical radius of its blocks, as seen in [23].

Lemma 1. Let Λ_1, Λ_2 be two bounded linear operators on $\mathcal{H}$. Then
(a) $w\left(\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}\right) = \max\{w(\Lambda_1), w(\Lambda_2)\}$; (b) $w\left(\begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2 & \Lambda_1 \end{bmatrix}\right) = \max\{w(\Lambda_1+\Lambda_2), w(\Lambda_1-\Lambda_2)\}$.

In particular,

$$w\left(\begin{bmatrix} 0 & \Lambda_2 \\ \Lambda_2 & 0 \end{bmatrix}\right) = w(\Lambda_2).$$

A special case of the mixed Schwarz inequality, which is found in [18], is the lemma that follows.

Lemma 2. Let Λ be a bounded linear operator on $\mathcal{H}$. Then, for any $x, y \in \mathcal{H}$, we have

$$|\langle \Lambda x, y\rangle|^2 \leq \langle |\Lambda| \, x, x\rangle \, \langle |\Lambda^*| \, y, y\rangle .$$

The following lemma is one of the most important results about the numerical radius that we will use in our proofs. This lemma can be found in [19].

Lemma 3. Let Λ be a bounded linear operator on $\mathcal{H}$. Then, $\quad w(\Lambda) = \max_{\theta\in\mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}\Lambda\right)\right\| = \max_{\theta\in\mathbb{R}} \left\|\operatorname{Im}\left(e^{i\theta}\Lambda\right)\right\|.$

The next lemma is the Buzano inequality (see [27]).

Lemma 4. Let $a, b, e \in \mathcal{H}$ with $\|e\| = 1$. Then $\quad |\langle a, e\rangle\langle e, b\rangle| \leq \frac{1}{2}\left(\|a\| \, \|b\| + |\langle a, b\rangle|\right).$

As stated in [29], Theorem 7 and Theorem 12 can be proved using the Kittaneh result, which is the subject of the following lemma.

Lemma 5. Let A, B be positive bounded linear operators on $\mathcal{H}$. Then

$$\|A+B\| \leq \max\{||A||, ||B||\} + \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|.$$

For $A \in \mathcal{B}(\mathcal{H})$, the spectral radius is defined as $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. Before concluding this section, we introduce the following lemma, which provides two important properties for $r(A)$ and is key to the proof of our first result.

Lemma 6. Let A, B be bounded linear operators on $\mathcal{H}$. Then
(a) $r(A) \leq w(A) \leq ||A||$ (the equality holds when A is normal), (b) $r(AB) = r(BA)$.

3. THE MAIN RESULTS

A new upper bound for the numerical radius of a $n \times n$ operator matrix is introduced by the following theorem, which we use to start our results.

Theorem 7. Let $A_1, A_2, ..., A_n$ be bounded operators on a complex Hilbert space $\mathcal{H}$, and let $M = [m_{ij}]_{n\times n}$ where

$$m_{ij} = \begin{cases} A_i, & i+j = n+1 \\ 0, & \text{otherwise} \end{cases}.$$

Then

$$w(M) \leq \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{2}\max_{1\leq i\leq n}\left\||A_{n-i+1}|^{\frac{1}{2}}|A_i^*|^{\frac{1}{2}}\right\| = \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{2}\max_{1\leq i\leq n} r^{\frac{1}{2}}\left(|A_{n-i+1}| \, |A_i^*|\right).$$

We start by noting that $|M| = [p_{ij}]_{n\times n}$ and $|M^*| = [q_{ij}]_{n\times n}$ where

$$p_{ij} = \begin{cases} |A_i|, & i = j \\ 0, & \text{otherwise} \end{cases}, \quad \text{and} \quad q_{ij} = \begin{cases} |A_i^*|, & i = j \\ 0, & \text{otherwise} \end{cases}.$$

Now, for any unit vector $y \in \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}$, we have

$$\begin{aligned}|\langle My, y\rangle| &\leq \langle |M|y,y\rangle^{\frac{1}{2}}\langle |M^*|y,y\rangle^{\frac{1}{2}} \leq \frac{1}{2}\langle (|M|+|M^*|)\,y,y\rangle \\ &= \frac{1}{2}\left\langle [p_{ij}+q_{ij}]_{n\times n}y,y\right\rangle \leq \frac{1}{2}w\left([p_{ij}+q_{ij}]_{n\times n}\right) \\ &= \frac{1}{2}\max_{1\leq i\leq n}\||A_{n-i+1}|+|A_i^*|\| \\ &= \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{2}\max_{1\leq i\leq n}\left\||A_{n-i+1}|^{\frac{1}{2}}|A_i^*|^{\frac{1}{2}}\right\|. \text{ (by Lemma 5)}\end{aligned}$$

Therefore,

$$\begin{aligned}w(M) &= \sup_{\|y\|=1}|\langle My,y\rangle| \leq \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{2}\max_{1\leq i\leq n}\left\||A_{n-i+1}|^{\frac{1}{2}}|A_i^*|^{\frac{1}{2}}\right\| \\ &= \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{2}\max_{1\leq i\leq n}\left\|\left(|A_{n-i+1}|^{\frac{1}{2}}|A_i^*|^{\frac{1}{2}}\right)\left(|A_{n-i+1}|^{\frac{1}{2}}|A_i^*|^{\frac{1}{2}}\right)^*\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{2}\max_{1\leq i\leq n}\left\||A_{n-i+1}|^{\frac{1}{2}}|A_i^*||A_{n-i+1}|^{\frac{1}{2}}\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{2}\max_{1\leq i\leq n}r^{\frac{1}{2}}\left(|A_{n-i+1}|^{\frac{1}{2}}|A_i^*||A_{n-i+1}|^{\frac{1}{2}}\right) \text{ (by Lemma 6(a))} \\ &= \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{2}\max_{1\leq i\leq n}r^{\frac{1}{2}}\left(|A_{n-i+1}||A_i^*|\right). \text{ (by Lemma 6(b))}\end{aligned}$$

This completes the proof of the theorem.

It is worth mentioning that Theorem 7 generalizes the result found in [15] for the case $n = 2$. Additionally, when $A_1 = A_2 = ... = A_n = A$, we get

$$w(A) \leq \frac{1}{2}\|A\| + \frac{1}{2}r^{\frac{1}{2}}\left(|A||A^*|\right). \tag{3}$$

It is evident that the upper bound in (3) is tighter than the upper bound in (1). For example, if we consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then then the upper bounds of (1) and (2) are 1 and $\frac{1}{\sqrt{2}}$, respectively. While the upper bound of (3) is $\frac{1}{2}$, which emphasize our claim . Next, we provide four lower bounds for $w(M)$, where M is an $n \times n$ operator matrix.

Theorem 8. Let $\{A_i\}_{i=1}^n \subseteq \mathcal{B}(\mathcal{H})$ and let M be as in Theorem 7. Then

$$w(M) \geq \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{4}\max_{1\leq i\leq n}\left|\left\|A_i + A_{n-i+1}^*\right\| - \left\|A_i - A_{n-i+1}^*\right\|\right|.$$

By Lemma 3, we have

$$w(M) \geq \|\mathrm{Re}(M)\| = \frac{1}{2}\max_{1\leq i\leq n}\left\|A_i + A_{n-i+1}^*\right\|, \quad w(M) \geq \|\mathrm{Im}(M)\| = \frac{1}{2}\max_{1\leq i\leq n}\left\|A_i - A_{n-i+1}^*\right\|.$$

Thus, for each $k \in \{1, 2, ..., n\}$ we have that

$$\begin{aligned}w(M) &\geq \frac{1}{2}\max\left\{\left\|A_k + A_{n-k+1}^*\right\|, \left\|A_k - A_{n-k+1}^*\right\|\right\} \\ &= \frac{1}{4}\left(\left\|A_k + A_{n-k+1}^*\right\| + \left\|A_k - A_{n-k+1}^*\right\|\right) + \frac{1}{4}\left|\left\|A_k + A_{n-k+1}^*\right\| - \left\|A_k - A_{n-k+1}^*\right\|\right| \\ &\geq \frac{1}{4}\left\|\left(A_k + A_{n-k+1}^*\right) \mp \left(A_k - A_{n-k+1}^*\right)\right\| + \frac{1}{4}\left|\left\|A_k + A_{n-k+1}^*\right\| - \left\|A_k - A_{n-k+1}^*\right\|\right|.\end{aligned}$$

This implies that $w(M) \geq \frac{1}{2}\max_{1\leq i\leq n}\|A_i\| + \frac{1}{4}\max_{1\leq i\leq n}\left|\left\|A_i + A_{n-i+1}^*\right\| - \left\|A_i - A_{n-i+1}^*\right\|\right|$. The following refinement of inequality (1) is a direct result of Theorem 8.

Corollary 9. Let $A \in \mathcal{B}(\mathcal{H})$. Then $w(A) \geq \frac{1}{2}\|A\| + \frac{1}{4}\left|\|A + A^*\| - \|A - A^*\|\right| \geq \frac{1}{2}\|A\|$, where $\frac{1}{2}\|A\|$ is the lower bound of (1).

Theorem 10. Let $\{A_i\}_{i=1}^n \subseteq \mathcal{B}(\mathcal{H})$ and let M be as in Theorem 7. Then

$$w^2(M) \geq \frac{1}{4}\max_{1\leq i\leq n}\left\||A_i|^2 + |A^*_{n-i+1}|^2\right\| \ + \frac{1}{8}\max_{1\leq i\leq n}\left|\left\|A_i + A^*_{n-i+1}\right\|^2 - \left\|A_i - A^*_{n-i+1}\right\|^2\right|.$$

For each $k \in \{1, 2, ..., n\}$, we have that

$$w(M) \geq \frac{1}{2}\max\left\{\left\|A_k + A^*_{n-k+1}\right\|, \left\|A_k - A^*_{n-k+1}\right\|\right\}.$$

Therefore,

$$\begin{aligned} w^2(M) &\geq \frac{1}{4}\max\left\{\left\|A_k + A^*_{n-k+1}\right\|^2, \left\|A_k - A^*_{n-k+1}\right\|^2\right\} \\ &= \frac{1}{8}\left(\left\|A_k + A^*_{n-k+1}\right\|^2 + \left\|A_k - A^*_{n-k+1}\right\|^2\right) \\ &\quad + \frac{1}{8}\left|\left\|A_k + A^*_{n-k+1}\right\|^2 - \left\|A_k - A^*_{n-k+1}\right\|^2\right|. \end{aligned}$$

This implies that

$$\begin{aligned} w^2(M) &\geq \frac{1}{2}\left(\|\mathrm{Re}(M)\|^2 + \|\mathrm{Im}(M)\|^2\right) + \frac{1}{8}\max_{1\leq i\leq n}\left|\left\|A_i + A^*_{n-i+1}\right\|^2 - \left\|A_i - A^*_{n-i+1}\right\|^2\right| \\ &= \frac{1}{2}\left(\left\|\mathrm{Re}^2(M)\right\| + \left\|\mathrm{Im}^2(M)\right\|\right) + \frac{1}{8}\max_{1\leq i\leq n}\left|\left\|A_i + A^*_{n-i+1}\right\|^2 - \left\|A_i - A^*_{n-i+1}\right\|^2\right| \\ &\geq \frac{1}{2}\left\|\mathrm{Re}^2(M) + \mathrm{Im}^2(M)\right\| + \frac{1}{8}\max_{1\leq i\leq n}\left|\left\|A_i + A^*_{n-i+1}\right\|^2 - \left\|A_i - A^*_{n-i+1}\right\|^2\right| \\ &= \frac{1}{4}\max_{1\leq i\leq n}\left\||A_i|^2 + |A^*_{n-i+1}|^2\right\| + \frac{1}{8}\max_{1\leq i\leq n}\left|\left\|A_i + A^*_{n-i+1}\right\|^2 - \left\|A_i - A^*_{n-i+1}\right\|^2\right|. \end{aligned}$$

Corollary 11. If $A \in \mathcal{B}(\mathcal{H})$, then

$$w^2(A) \geq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{8}\left|\|A + A^*\|^2 - \|A - A^*\|^2\right| \geq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|,$$

where $\frac{1}{4}\left\||A|^2 + |A^*|^2\right\|$ is the lower bound of (2).

Theorem 12. Let $\{\Lambda_i\}_{i=1}^n \subseteq \mathcal{B}(\mathcal{H})$ and let T be as in Theorem 7. Then,

$$\begin{aligned} w^2(T) &\geq \frac{1}{8}\max\left\{\max_{1\leq i\leq n}\left\|\Lambda_i + \Lambda^*_{n-i+1}\right\|^2, \max_{1\leq i\leq n}\left\|\Lambda_i - \Lambda^*_{n-i+1}\right\|^2\right\} \\ &\quad + \frac{1}{8}\max_{1\leq i\leq n}\left\|\Lambda_i + \Lambda^*_{n-i+1}\right\|\max_{1\leq i\leq n}\left\|\Lambda_i - \Lambda^*_{n-i+1}\right\| \\ &\geq \frac{1}{4}\max_{1\leq i\leq n}\left\||\Lambda_i|^2 + |\Lambda^*_{n-i+1}|^2\right\|. \end{aligned}$$

For the first inequality,

$$\begin{aligned} w^2(T) &= \frac{1}{2}w^2(T) + \frac{1}{2}w^2(T) \geq \frac{1}{2}\max\left\{\|\mathrm{Re}(T)\|^2, \|\mathrm{Im}(T)\|^2\right\} + \frac{1}{2}\|\mathrm{Re}(T)\|\,\|\mathrm{Im}(T)\| \\ &= \frac{1}{8}\max\left\{\max_{1\leq i\leq n}\left\|\Lambda_i + \Lambda^*_{n-i+1}\right\|^2, \max_{1\leq i\leq n}\left\|\Lambda_i - \Lambda^*_{n-i+1}\right\|^2\right\} \\ &\quad + \frac{1}{8}\max_{1\leq i\leq n}\left\|\Lambda_i + \Lambda^*_{n-i+1}\right\|\max_{1\leq i\leq n}\left\|\Lambda_i - \Lambda^*_{n-i+1}\right\|. \end{aligned}$$

For the second inequality,

$$\begin{aligned}\frac{1}{4}\max_{1\le i\le n}\left\||\Lambda_i|^2+|\Lambda^*_{n-i+1}|^2\right\| &= \frac{1}{2}\left\|\mathrm{Re}^2(T)+\mathrm{Im}^2(T)\right\| \\ &\le \frac{1}{2}\max\left\{\left\|\mathrm{Re}^2(T)\right\|,\left\|\mathrm{Im}^2(T)\right\|\right\}+\frac{1}{2}\left\|\mathrm{Re}(T)\right\|\left\|\mathrm{Im}(T)\right\| \\ &= \frac{1}{8}\max\left\{\max_{1\le i\le n}\left\|\Lambda_i+\Lambda^*_{n-i+1}\right\|^2,\max_{1\le i\le n}\left\|\Lambda_i-\Lambda^*_{n-i+1}\right\|^2\right\} \\ &\quad+\frac{1}{8}\max_{1\le i\le n}\left\|\Lambda_i+\Lambda^*_{n-i+1}\right\|\max_{1\le i\le n}\left\|\Lambda_i-\Lambda^*_{n-i+1}\right\|.\end{aligned}$$

By Theorem 12 and Lemma 1(b), we have the following corollary.

Corollary 13. Let $\Lambda\in\mathcal{B}(\mathcal{H})$. Then

$$w^2(\Lambda)\ge\frac{1}{8}\max\left\{\|\Lambda+\Lambda^*\|,\|\Lambda-\Lambda^*\|\right\}+\frac{1}{8}\|\Lambda+\Lambda^*\|\|\Lambda-\Lambda^*\|\ge\frac{1}{4}\left\||\Lambda|^2+|\Lambda^*|^2\right\|,$$

where $\frac{1}{4}\left\||\Lambda|^2+|\Lambda^*|^2\right\|$ is the lower bound of (2).

To prove our next result, we need the following lemma, which can be found in [28].

Lemma 14. Let $A,B\in\mathcal{B}(\mathcal{H})$. Then

$$\|A+B\|^2\le 2\max\left\{|||A|^2+|B|^2||,|||A^*|^2+|B^*|^2||\right\}.$$

Theorem 15. Let $\{\Lambda_i\}_{i=1}^n\subseteq\mathcal{B}(\mathcal{H})$ and let T be as in Theorem 7. Then

$$w^4(T)\ge\frac{1}{32}\max_{1\le i\le n}\left\|\Lambda_i+\Lambda^*_{n-i+1}\right\|^4+\frac{1}{32}\max_{1\le i\le n}\left\|\Lambda_i-\Lambda^*_{n-i+1}\right\|^4\ge\frac{1}{16}\max_{1\le i\le n}\left\||\Lambda^*_i|^2+|\Lambda_{n-i+1}|\right\|^2.$$

For the first inequality, we have

$$\begin{aligned}w^4(T)&\ge\max\left\{\|\mathrm{Re}(T)\|^4,\|\mathrm{Im}(T)\|^4\right\}\ge\frac{1}{2}\left(\|\mathrm{Re}(T)\|^4+\|\mathrm{Im}(T)\|^4\right)\\&=\frac{1}{32}\max_{1\le i\le n}\left\|\Lambda_i+\Lambda^*_{n-i+1}\right\|^4+\frac{1}{32}\max_{1\le i\le n}\left\|\Lambda_i-\Lambda^*_{n-i+1}\right\|^4.\end{aligned}$$

Now, for the second inequality, we have

$$\begin{aligned}\frac{1}{16}\max_{1\le i\le n}\left\||\Lambda_i{}^*|^2+|\Lambda_{n-i+1}|^2\right\|^2&=\frac{1}{4}\left\|\mathrm{Re}^2(T)+\mathrm{Im}^2(T)\right\|^2\\&\le\frac{1}{2}\left(\left\|\mathrm{Re}^4(T)+\mathrm{Im}^4(T)\right\|\right)\quad\text{(by Lemma 14)}\\&\le\frac{1}{2}\left(\|\mathrm{Re}(T)\|^4+\|\mathrm{Im}(T)\|^4\right)\\&=\frac{1}{32}\max_{1\le i\le n}\left\|\Lambda_i+\Lambda^*_{n-i+1}\right\|^4+\frac{1}{32}\max_{1\le i\le n}\left\|\Lambda_i-\Lambda^*_{n-i+1}\right\|^4.\end{aligned}$$

Remark 1. Let $\Lambda_1=\Lambda_2=...=\Lambda_n=\Lambda$. Then by Theorem 15, we have

$$w^2(\Lambda)\ge\frac{1}{4\sqrt{2}}\sqrt{\|\Lambda+\Lambda^*\|^4+\|\Lambda-\Lambda^*\|^4}\ge\frac{1}{4}\left\||\Lambda|^2+|\Lambda^*|^2\right\|,$$

where $\frac{1}{4}\left\||\Lambda|^2+|\Lambda^*|^2\right\|$ is the lower bound of (2).

At the end of this paper, we remark that all the inequalities in our results become equalities if $\Lambda_1=\Lambda$, where Λ is a bounded linear operator, and $\Lambda_2=\Lambda_3=...=\Lambda_n=0$.

4. CONCLUSION

In this paper, we have introduced several new inequalities that help limit the Euclidean numerical radius and its arithmetic operations. These results also provide useful tools for establishing inequalities for the numerical radius $w(T)$ of block operators. The study offers a new inequality that provides more accurate bounds for the numerical radius by carefully analyzing inequalities for numerical radii in $n \times n$ operator matrices that include block operators. Additionally, we show that the bounds we obtained here not only enhance but also generalize some of the existing lower and upper bounds. This analysis emphasizes the significance of understanding bounds in matrices and linear operators, and it highlights the key role that symmetry plays in mathematics across various disciplines.

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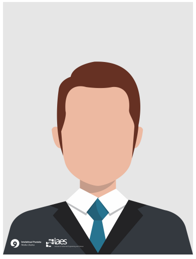

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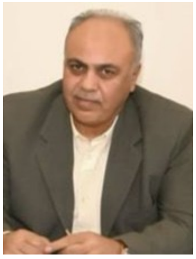

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