An efficient approximate method for solving Bratu's boundary value problem

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ABSTRACT

We compute the numerical solution of Bratu's boundary value problem (BVP). To achieve this, we apply a new and useful approach to solve Bratu's boundary value problem by using Green's function and a new integral operator, along with a modified version of the Adomian decomposition method. This process produces solutions that call for the boundary conditions to be applied explicitly. Statistical results demonstrating the robustness and efficiency of the proposed scheme are included. An exact and approximate solution comparison is made with known results. The quantitative outcomes showcase our novel approach's high numerical precision and consistency across a range of parameter configurations.

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1. INTRODUCTION

Ordinary differential equations that are nonlinear are the result of mathematical modeling of various physical systems. To analyze mathematical modeling that yields solutions that match physical reality (that is, the real world of physics) an efficient method is needed. As a result, we need to be able to solve strongly nonlinear ordinary differential equations. The Chandrasekhar model of universe expansion, fuel ignition in thermal combustion theory, and Bratu's problem, which was used to model a combustion problem in a numerical slab, are examples of specific models. It encourages a thermal reaction in a stiff material that depends on the equilibrium between heat produced chemically and heat transferred through conduction [1]–[4].

We examine the elliptic nonlinear partial differential equation with homogeneous Dirichlet boundary conditions at the boundary known as the classical Bratu problem [1]. The Bratu's model has been computationally handled by a number of methods, including the Laplace transform decomposition numerical algorithm [5], [6], differential transform method [7], weighted residual method [8], shooting method [9], finite difference method, finite element approximation, and Adomian decomposition method (ADM) [10], [11].

DTM [12] provide an effective numerical technique for approximating geometric boundary value problems. The hardest component of the Bratu's equation is the nonlinear term, e^y. Therefore, by applying the strategy covered in [12], Sinc geometric can readily solve Braut's equation. The variation iteration method is used by Anakira *et al.* [13] to examine and modify an approximate analytical solution for solving the fuzzy Bratu's equation. A numerical approach based on Chebyshev wavelet approximations for the one-dimensional Bratu's problem is presented in [14]. In [15], Bratu's boundary value problem is solved using a new integral operator by utilizing the adomian decomposition method and the restarted adomian decomposition method with new techniques. A hybrid Adomian decomposition approach and an additional integral transform called the Kashuri Fundo transform were used to solve Bratu's equation [16]. The primary goal of the numerical solution of the Bratu's boundary value problem on a Banach space setting is to incorporate a Green's function into a novel two-step iteration scheme, as demonstrated in [17]. A numerical solution for the two-dimensional Bratu's problem is obtained in [18] through the development of a new iterative finite difference algorithm. Conversely, they calculate two-branched numerical solutions for the nonlinear Bratu's problem in [19]. Bratu's problem has a closed-form solution that was created in [20]. Using artificial neural network architecture and a soft computing technique that makes use of the symbiotic organism search algorithm, an efficient meta-heuristic, better solutions for Bratu's differential equation are obtained in [21].

This paper shows that nonlinear second-order boundary value problems can be effectively solved numerically through manipulation of the Adomian decomposition method.

$$\frac{d^2y}{dx^2} = \lambda(x)e^{\mu y}, \;\; y(0) = y(1) = 0 \tag{1}$$

in which $\lambda(x) > 0$ may be an exponential, rational, or polynomial function, and $\mu > 0$ is a constant. Owing to the wide range of possibilities, we shall take $\lambda(x)$ to represent any analytic function of x with a power series expansion. Diffusion theory and celestial machines routinely use (1), for instance in mechanical problems where dissipation is absent.

2. BRATU'S BOUNDARY VALUE PROBLEM

This section presents a numerical algorithm to solve a general boundary value problem using the Adomian decomposition technique.

$$\frac{d^2y}{dx^2} = \lambda(x)F(y), \;\; y(0) = \alpha, \; y(1) = \beta. \tag{2}$$

where $F(y)$, the nonlinear function, can be developed in a series of the form $F(y) = \sum_{n=0}^{\infty} a_n y^n$ since it is assumed to be analytic. Equation (1) presents the same problem for us to discuss as a special case, if $F(y) = e^{\mu y}$. The basic concept is to use Green's function to determine the integral representation of (2), and then modify Adomian decomposition to produce the nonlinear integral equation of the following form.

$$y(x) = \int_0^1 \lambda(x)G(x,\xi)F(y(\xi))d\xi + (1-x)\alpha + x\beta. \tag{3}$$

The well known Green's function for (2) is given by,

$$G(x,\xi) = \begin{cases} \xi(1-x), & 0 \le \xi \le x \\ x(1-\xi), & x \le \xi \le 1. \end{cases}$$

ADM [22], [23] assumes a series solution for (2) of the form,

$$y(x) = \sum_{n=0}^{\infty} y_i(x) \tag{4}$$

The above infinite series converges if $\sum_{n=0}^{\infty}(1+\epsilon)^n|y_n| < \infty$, for some small number ϵ.Physical issues typically satisfy this requirement. Assume that the nonlinear function $F(y)$ has the Taylor expansion around y_0 as follows to demonstrate Adomian's scheme.

$$F(y) = F(y_0) + F'(y_0)(y-y_0) + \frac{1}{2!}F''(y_0)(y-y_0)^2 + ... \tag{5}$$

Substituting the difference $y - y_0 = y_1 + y_2 + ...$ from (4) into (5), to obtain,

$$F(y) = F(y_0) + F'(y_0)(y_1 + y_2 + ...) + \frac{1}{2!}F''(y_0)(y_1 + y_2 + ...)^2 + ...$$

After simplifying the above equation, we get,

$$F(y) = F(y_0) + F'(y_0)(y_1 + y_2 + ...) + \frac{1}{2!}F''(y_0)(y_1^2 + 2y_1y_2 + 2y_1y_3 + y_2^2 + 2y_2y_3 + y_3^2 + ...) \quad (6)$$

Now, reordering and rearranging of the terms in the above equation, we arrive at Adomian's polynomials as [24]: $A_0 = F(y_0),\ A_1 = y_1F'(y_0),\ A_2 = y_2F'(y_0) + \frac{1}{2!}y_1^2F''(y_0)$ and so on for more terms. Substituting these A_n's together with the series in (3), we arrive at,

$$y_0(x) + y_1(x) + y_2(x) + ... = \int_0^1 \lambda(x)G(x,\xi)(A_0 + A_1 + A_2 + ...)d\xi + (1-x)\alpha + x\beta.$$

If the series is convergent, then we determine each term of the series solution in (3) recursively as,

$$\begin{aligned} y_0(x) &= (1-x)\alpha + x\beta \\ y_1(x) &= \int_0^1 \lambda(x)G(x,\xi)A_0(y_0)d\xi \\ &\vdots \\ y_n(x) &= \int_0^1 \lambda(x)G(x,\xi)A_{n-1}(y_0, y_1, ..., y_{n-1})d\xi \end{aligned}$$

The above algorithm determines the y_is and hence determines the approximate solution $y_a(x) = y_0(x) + y_1(x) + ... + y_n = \lim_{M\to\infty}\sum_{n=0}^{M} y_n(x)$, where M is the number of terms that we found. It is possible to find an exact solution in closed form in some circumstances. Moreover, the solutions of decomposition series usually converge quickly. The convergence of the decomposition series has been studied by a number of writers [25], [26]. The discussion that was previously presented can be clearly summarized by looking at the numerical results below.

3. RESULTS AND DISCUSSION

Here, we report our numerical findings for Bratu's problem. To demonstrate the robustness of the schemes, we will investigate various parameter regimes for the eigenvalue λ. The calculations show how simple it is to assemble the method, run it, and select parameters. The numerical results demonstrate the convergence and accuracy of our approach.

Examine Bratu's boundary value problem.

$$y''(x) = -\lambda e^{y(x)},\ y(0) = y(1) = 0. \quad (7)$$

Bratu's equation appeared first in the theory of combustion, after that, the equation appeared in many mathematical models concerned with physical applications. Specifically, it contains the solid fuel ignition model for the thermal reaction process in a combustible, nondeformable material of constant density during the ignition period [2], and the Chandrashekhar model for the expansion of the universe [3]. As stated in [2], (7) has an exact solution provided by (8),

$$y(x) = -2\log\left[\frac{\cosh(0.5(x-0.5)\theta)}{\cosh(0.25\theta)}\right] \quad (8)$$

provided that θ is a solution of $\theta = \sqrt{2\lambda}\cosh(\theta/4)$. To find the critical value of λ, differentiating both sides we obtain $1 = \frac{1}{4}\sqrt{2\lambda}\sinh(\theta/4)$, in which the critical value of λ is $\lambda_c = \frac{8}{\sinh^2(\theta/4)}$. It follows that when $\lambda > \lambda_c, \lambda = \lambda_c$ and $0 < \lambda < \lambda_c$, where $\lambda_c = 3.5138$ is the critical value, Bratu's problem has zero solution, a unique solution, and two bifurcated solutions, respectively. For the ADM solution, (7) can be written in an integral equation form as,

$$y(x) = \lambda \int_0^1 G(x,\xi)e^{y(\xi)}d\xi$$

where $G(x, \xi)$ is the Green's function. The nonlinear function e^y may be expanded as an infinite sum of Adomian polynomials, as $e^y = \sum_{i=0}^{\infty} A_i = A_0 + A_1 + A_2 + ...$, where $A_0 = 1,\ A_1 = y_1 e^{y_0},\ A_2 = y_2 + \frac{1}{2!}y_1^2,\ A_3 = y_3 + y_1 y_2 + \frac{1}{3!}y_1^3,\ A_4 = y_4 + y_1 y_3 + \frac{1}{2!}y_2^2 + \frac{1}{2}y_1^2 y_2 + \frac{1}{4!}y_1^4$. According to the analysis in the previous section, we start with $y_0(x) = 0$, the ADM solution of Bratu's equation can be written as:

$$y_{k+1}(x) = \lambda(1-x)\int_0^x \xi A_k(\xi)d\xi + \lambda x\int_x^1 (1-\xi)A_k(\xi)d\xi,\ k = 0, 1, 2...$$

Using Mathematica, the first five iterations in y are calculated, their sum is denoted by $y_a(x)$, we list the first two terms:

$$y_1(x) = \frac{\lambda x^2}{2}(1-x) + \lambda x(\frac{1}{2} - x + \frac{x^2}{2})$$

$$y_2(x) = \lambda(1-x)(\frac{\lambda x^3}{6} - \frac{\lambda x^4}{8}) + \lambda x(\frac{\lambda}{24} - \frac{\lambda x^2}{4} + \frac{\lambda x^3}{3} - \frac{\lambda x^4}{8}).$$

The approximation using five terms in equation is carried out for Bratu's model with $\lambda = 1$ at $x = 0.1, 0.2, ...,$ 0.9. Table 1 exhibits the results of the approximate solution, and the exact solution as given in the closed form (8). The mathematical formulation of electrospinning process is linked also to the famous Bratu's equation as given by (9),

$$\frac{d^2y}{dx^2} = \lambda(x)e^y,\quad y(0) = y(1) = 0 \tag{9}$$

where $y = -6\ln u$ and u represents the jet's axial velocity. After solving the mass, linear momentum, and electric charge balance equations, Wan *et al.* [27] calculated the value,

$$\lambda = \frac{18E^2}{\rho^2 r^4}(I - r^2 KE).$$

The parameters I, K, E, ρ, and r represent the jet's current, conductivity, voltage, density, and radius, respectively [28]. Three different selections of the parameters E, K, ρ, I, r were chosen such that they made the value of λ is equal to 1, sometimes less than 1, and sometimes greater than 1, and Figures 1 and 2 summarise our results. Numerical experiments indicate that different eigenvalues behave similarly [29]. It is interesting to point out that the solution is bounded for all values of its domain.

Table 1. ADM approximation for $y'' = \lambda e^y$ for $\lambda = 1$

x	Exact solution	ADM	Error
0.1	0.049846	0.049817	2.92373E-05
0.2	0.089189	0.089133	5.64677E-05
0.3	0.117609	0.117530	7.90006E-05
0.4	0.134790	0.134696	9.39895E-05
0.5	0.140539	0.140440	9.92575E-05
0.6	0.134790	0.134696	9.39895E-05
0.7	0.117609	0.117530	7.90006E-05
0.8	0.089189	0.089133	5.64677E-05
0.9	0.049846	0.049817	2.92373E-05

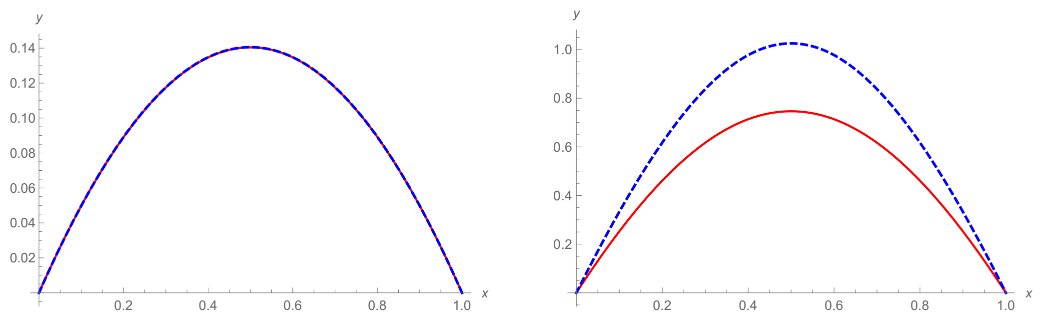
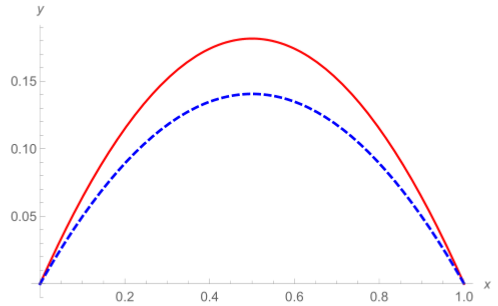


Figure 1. The ADM solution when $\lambda = 1.0$ (left) and (right) when $\lambda = 3.51$ for $y'' = \lambda e^y$



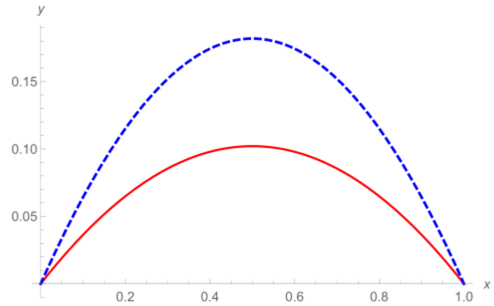


Figure 2. The ADM solution when $\lambda = 1.25$ (left) and (right) when $\lambda = 0.75$ for $y'' = \lambda e^y$

4. CONCLUSION

In order to obtain approximate analytical solutions for Bratu-type equations, this paper successfully applies the ADM. Without linearization or restrictive presumptions, the method was used directly. ADM is convenient and effective, as demonstrated by comparisons to the precise solution. For a wide range of linear and nonlinear differential equations, ADM is useful in providing both analytical and numerical solutions. For physical problems, ADM provides fast-convergent, realistic series solutions. The outcomes presented in this paper show that, with very little computational effort, our method can be used to obtain precise numerical solutions of nonlinear boundary value problems. In a reexamination, the Bratu's problem can be solved by substituting the fractional derivative of order α for the second derivative, where $1 < \alpha \leq 2$. In the event that the derivative in (1) is marginally less than 2, this will allow the eigenvalue behavior to be determined.

REFERENCES

[1] G. Bratu, "On nonlinear integral equations (in France)," *Bulletin de la Société Mathématique de France*, vol. 42, pp. 113–142, 1914.
[2] J. Jacobsen and K. Schmitt, "The Liouville–Bratu–gelfand problem for radial operators," *Journal of Differential Equations*, vol. 184, no. 1, pp. 283–298, Sep. 2002, doi: 10.1006/jdeq.2001.4151.
[3] S. Chandrasekhar, *An introduction to the study of stellar structure*. Dover Publications, 1957.
[4] M. Baccouch and H. Temimi, "A new derivation of the closed-form solution of Bratu's problem," *International Journal of Applied and Computational Mathematics*, vol. 9, no. 5, Oct. 2023, doi: 10.1007/s40819-023-01570-y.
[5] S. A. Khuri, "A new approach to Bratu's problem," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 131–136, 2004.
[6] E. Deeba, S. A. Khuri, and S. Xie, "An algorithm for solving boundary value problems," *Journal of Computational Physics*, vol. 159, no. 2, pp. 125–138, Apr. 2000, doi: 10.1006/jcph.2000.6452.
[7] J. M. W. Munganga, J. N. Mwambakana, R. Maritz, T. A. Batubenge, and G. M. Moremedi, "Introduction of the differential transform method to solve differential equations at undergraduate level," *International Journal of Mathematical Education in Science and Technology*, vol. 45, no. 5, pp. 781–794, Jul. 2014, doi: 10.1080/0020739X.2013.877609.
[8] Y. A. S. Aregbesola, "Numerical solution of Bratu problem using the method of weighted residual," *Electronic Journal of the South African Mathematical Sciences*, vol. 3, no. 1, pp. 1–7, 2003.
[9] U. M. Ascher, R. M. M. Mattheij, and R. D. Russell, *Numerical solution of boundary value problems for ordinary differential equations*. Society for Industrial and Applied Mathematics (SIAM), 1995.
[10] A.-M. Wazwaz, "Adomian decomposition method for a reliable treatment of the Bratu-type equations," *Applied Mathematics and Computation*, vol. 166, no. 3, pp. 652–663, Jul. 2005, doi: 10.1016/j.amc.2004.06.059.
[11] A. M. Wazwaz, "The numerical solution of special fourth-order boundary value problems by the modified decomposition method," *International Journal of Computer Mathematics*, vol. 79, no. 3, pp. 345–356, 2002, doi: 10.1080/00207160211928.
[12] A. Darweesh, K. Al-Khaled, and M. Algamara, "Solving boundary value problems by sinc method and geometric sinc method," *Symmetry*, vol. 16, no. 4, Apr. 2024, doi: 10.3390/sym16040411.
[13] N. R. Anakira, A. H. Shather, A. F. Jameel, A. K. Alomari, and A. Saaban, "Direct solution of uncertain bratu initial value problem," *International Journal of Electrical and Computer Engineering (IJECE)*, vol. 9, no. 6, pp. 5075–5083, Dec. 2019, doi: 10.11591/ijece.v9i6.pp5075-5083.
[14] C. Yang and J. Hou, "Chebyshev wavelets method for solving Bratu's problem," *Boundary Value Problems*, vol. 2013, no. 1, Dec. 2013, doi: 10.1186/1687-2770-2013-142.
[15] M. Al-Mazmumy, A. Al-Mutairi, and K. Al-Zahrani, "an efficient decomposition method for solving Bratu's boundary value problem," *American Journal of Computational Mathematics*, vol. 07, no. 01, pp. 84–93, 2017, doi: 10.4236/ajcm.2017.71007.
[16] H. A. Peker and F. A. Çuha, "Solving one-dimensional Bratu'S Problem via kashuri fundo decomposition method," *Romanian Journal of Physics*, vol. 68, no. 5–6, 2023.
[17] J. Ahmad, M. Arshad, K. Ullah, and Z. Ma, "Numerical solution of Bratu's boundary value problem based on Green's function and a novel iterative scheme," *Boundary Value Problems*, vol. 2023, no. 1, Oct. 2023, doi: 10.1186/s13661-023-01791-6.
[18] H. Temimi, M. Ben-Romdhane, M. Baccouch, and M. O. Musa, "A two-branched numerical solution of the two-dimensional Bratu's problem," *Applied Numerical Mathematics*, vol. 153, pp. 202–216, 2020, doi: 10.1016/j.apnum.2020.02.010.

[19] H. Temimi and M. Ben-Romdhane, "A highly accurate discontinuous Galerkin method for solving nonlinear Bratu's problem," *Alexandria Engineering Journal*, vol. 95, pp. 50–58, May 2024, doi: 10.1016/j.aej.2024.03.072.

[20] M. Baccouch and H. Temimi, "A new derivation of the closed-form solution of Bratu's problem," *International Journal of Applied and Computational Mathematics*, vol. 9, no. 5, Oct. 2023, doi: 10.1007/s40819-023-01570-y.

[21] A. Ahmad, M. Sulaiman, A. J. Aljohani, A. Alhindi, and H. Alrabaiah, "Design of an efficient algorithm for solution of Bratu differential equations," *Ain Shams Engineering Journal*, vol. 12, no. 2, pp. 2211–2225, Jun. 2021, doi: 10.1016/j.asej.2020.11.007.

[22] G. Adomian, "A review of the decomposition method in applied mathematics," *Journal of Mathematical Analysis and Applications*, vol. 135, no. 2, pp. 501–544, 1988, doi: 10.1016/0022-247X(88)90170-9.

[23] G. Adomian, *Solving frontier problems of physics: the decomposition method*. Dordrecht: Springer Netherlands, 1994.

[24] A. M. Wazwaz, "A new algorithm for calculating adomian polynomials for nonlinear operators," *Applied Mathematics and Computation*, vol. 111, no. 1, pp. 33–51, 2000, doi: 10.1016/s0096-3003(99)00063-6.

[25] Y. Cherruault, "Convergence of Adomian's method," *Mathematical and Computer Modelling*, vol. 14, pp. 83–86, 1990, doi: 10.1016/0895-7177(90)90152-D.

[26] Y. Cherruault and G. Adomian, "Decomposition methods: A new proof of convergence," *Mathematical and Computer Modelling*, vol. 18, no. 12, pp. 103–106, 1993, doi: 10.1016/0895-7177(93)90233-O.

[27] Y. Q. Wan, Q. Guo, and N. Pan, "Thermo-electro-hydrodynamic model for electrospinning process," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 5, no. 1, pp. 5–8, 2004, doi: 10.1515/IJNSNS.2004.5.1.5.

[28] H. Q. Kafri, S. A. Khuri, and A. Sayfy, "Bratu-like equation arising in electrospinning process: a Green's function fixed-point iteration approach," *International Journal of Computing Science and Mathematics*, vol. 8, no. 4, 2017, doi: 10.1504/IJCSM.2017.085869.

[29] V. Goyal, "On a second order nonlinear boundary value problem," *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1406–1408, 2006, doi: 10.1016/j.aml.2006.02.012.

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