

Fuzzy linear programming with the intuitionistic polygonal fuzzy numbers

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Article Info

Article history:

Received Jul 28, 2023

Revised Dec 18, 2023

Accepted Jan 5, 2024

Keywords:

Fuzzy numbers

Intuitionistic fully fuzzy linear programming

Intuitionistic fuzzy numbers

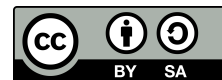
Linear programming

Simplex method

ABSTRACT

In real world applications, data are subject to ambiguity due to several factors; fuzzy sets and fuzzy numbers propose a great tool to model such ambiguity. In case of hesitation, the complement of a membership value in fuzzy numbers can be different from the non-membership value, in which case we can model using intuitionistic fuzzy numbers as they provide flexibility by defining both a membership and a non-membership functions. In this article, we consider the intuitionistic fuzzy linear programming problem with intuitionistic polygonal fuzzy numbers, which is a generalization of the previous polygonal fuzzy numbers found in the literature. We present a modification of the simplex method that can be used to solve any general intuitionistic fuzzy linear programming problem after approximating the problem by an intuitionistic polygonal fuzzy number with n edges. This method is given in a simple tableau formulation, and then applied on numerical examples for clarity.

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1. INTRODUCTION

Linear programming (LP) is a widely used tool in operations research since many optimization problems can be expressed or simplified to a linear format. When full information about data and parameters are known, traditional methods can be used for modelling and solving the problem. However, in many applications, data are subject to vagueness or impreciseness; for example the prices may depend on the quantity or the day in the year. In 1965, Zadeh [1] introduced the concept of fuzzy logic to formulate the vagueness of data. Fuzzy numbers are represented by a membership function that represents the degree of membership of a number to a given set. The shape of the membership function determines the type of fuzzy number such as the triangular, trapezoidal, hexagonal and polygonal shapes. The fuzzy logic has been applied in many real life applications and tens of journals has been established that deal with fuzzy logic and its applications. To give some, Singh *et al.* [2] counted tens of thousands that deal with applications of fuzzy logic. Recent applications include data acquisition can be found in Haddin *et al.* [3], in communication system [4], in grid connected PV inverter [5], in Ambulance detection for smart traffic light applications [6] and many much more.

Bellman and Zadeh [7] are the first to introduce fuzzy environments in decision making, since then many publications have suggested methods for solving linear programming in the fuzzy environment. In particular, fully fuzzy linear programming (FFLP) problems with triangular, trapezoidal and hexagonal fuzzy numbers have been considered by several authors. For instance, Kumar and Kaur [8] considered the triangular

FFLP and Das *et al.* [9] considered the trapezoidal FFLP. Recently, Tuffaha and Alrefaei [10] presented a polygonal FFLP which is a generalization of all the above FFLP.

In many cases, the complement of a membership function on a fuzzy set does not mean a non-membership, which happens in the case of hesitation. Therefore, the non-membership degree can be introduced by another function called the non-membership function, which is the main character of intuitionistic fuzzy numbers [11]. Solving optimization problems in intuitionistic fuzzy environment has been introduced by Angelov [12] by maximizing the degree of membership and minimizing the degree of non-membership then converting the problem to a crisp (unfuzzy) LP. Many solution methods have been presented for solving special cases of IFLP such as the triangular and trapezoidal intuitionistic fuzzy linear programming (IFLP) problems. Converting to easily-solvable crisp versions of the problem is common by using either ranking functions directly [13] or by dividing the problem based on (α, β) -cuts and then using a ranking function [14]. After the conversion to crisp problems, the crisp simpler versions are ready to be solved by conventional methods such as the simplex algorithm. Suresh *et al.* [15] suggested such a method, but Sidhu and Kumar [16] later showed that the ranking function presented by the authors is incorrect; they fixed it and resolved the IFLP using the corrected ranking function. On the other hand, a single step algorithm that directly solves the problem without converting it to crisp was also suggested by Nagoorgani and Ponnalagu [17]. Discussion of duality of the IFLP problem can also be found in [14], and a dual simplex method was proposed by Goli and Nasserli [18]. A fully fuzzy IFLP with unconstrained LR-type of intuitionistic fuzzy numbers was introduced by Singh and Yadav [19]. Their method resembles the Mehar's method proposed by Kaur and Kumar [20] for solving fully fuzzy linear programming where they use the (α, β) -cut and define a new product on the LR-type of intuitionistic fuzzy numbers and use it to solve the problem. Recently, Malik *et al.* [21] present an approach for solving a fully IFLP with unrestricted decision variables. A multi-objective IFLP for solving a closed loop supply chain has been presented by Kousar *et al.* [22], while Singh and Yadav [23] solve a multi-objective IFLP using various membership functions.

Most of the above articles are used for solving special cases of intuitionistic linear programming problems with triangular, trapezoidal or hexagonal intuitionistic fuzzy numbers and so on. Moreover, most of the above methods lack to provide general definitions of the binary operations between two intuitionistic fuzzy numbers. Most of the previous binary operations were defined for specific cases such as triangular, trapezoidal or hexagonal. Moreover, the product of two intuitionistic fuzzy numbers lack to preserve some known properties such as the ranking of fuzzy numbers.

In this paper, we consider the intuitionistic fuzzy linear programming problem (IFLP) problem with general intuitionistic fuzzy numbers. This intuitionistic fuzzy numbers can be approximated by an polygonal intuitionistic fuzzy numbers with n edges (n -IPFN) and the simplex method is modified for solving this problem using the binary operations presented by Alrefaei and Tuffaha [24]. Alrefaei and Tuffaha [24] provide a generalized definition of the product between intuitionistic polygonal fuzzy number with n edges (n -IPFN) that can preserve all the known properties such as the ranking of intuitionistic fuzzy numbers. These binary operations can be used in all types of the known intuitionistic fuzzy numbers such as triangular, trapezoidal and hexagonal fuzzy numbers because they are considered as special cases of n -IPFN.

This paper is organized as follows: section 2 previews some preliminaries about the intuitionistic fuzzy numbers and give the definition of the polygonal intuitionistic fuzzy numbers. In section 3, we present the intuitionistic fully fuzzy linear programming (IFFLP) with n -IPFN, in section 4, we present the intuitionistic fuzzy simplex method and implement it into two numerical examples. Finally concluding remarks are presented in section 5.

2. PRELIMINARY

We first give some definitions about the intuitionistic fuzzy sets and numbers and the definition of intuitionistic polygonal fuzzy number (IPFN- n).

Definition 1. [11] An intuitionistic fuzzy set (IFS) \tilde{A}^I is a triple $\langle X, \mu_{\tilde{A}^I}, \nu_{\tilde{A}^I} \rangle$, where X is any set and $\mu_{\tilde{A}^I}, \nu_{\tilde{A}^I} : X \rightarrow [0, 1]$ are called the membership and non-membership functions, respectively. The complementary of this sum to 1 is called the degree of hesitation, $\pi_{\tilde{A}^I}(x) = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x) \in [0, 1]$.

Definition 2. [25] A real intuitionistic fuzzy number (IFN), \tilde{A}^I , is an intuitionistic fuzzy subset of \mathbb{R} with a membership $\mu_{\tilde{A}^I}$ and a non-membership $\nu_{\tilde{A}^I}$ functions, which can be described as:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \mu_{\tilde{A}^I}^L(x) & ; a < x \leq b \\ 1 & ; b < x \leq c \\ \mu_{\tilde{A}^I}^R(x) & ; c < x \leq d \\ 0 & \text{otherwise} \end{cases}, \nu_{\tilde{A}^I}(x) = \begin{cases} \nu_{\tilde{A}^I}^L(x) & ; a' < x \leq b' \\ 0 & ; b' < x \leq c' \\ \nu_{\tilde{A}^I}^R(x) & ; c' < x \leq d' \\ 1 & \text{otherwise} \end{cases}$$

where a, b, c, d, a', b', c' and d' are real numbers, $\mu_{\tilde{A}^I}^L$ is increasing function, $\mu_{\tilde{A}^I}^R$ is decreasing function, $\nu_{\tilde{A}^I}^L$ is decreasing function and $\nu_{\tilde{A}^I}^R$ is increasing function.

Now we have the definition of the intuitionistic polygonal fuzzy number.

Definition 3. [24] Let $\{(a_0, a_1, \dots, a_n; b_0, b_1, \dots, b_n), (a'_0, a'_1, \dots, a'_n; b'_0, b'_1, \dots, b'_n)\}$ be real numbers, an intuitionistic polygonal fuzzy number (n -IPFN) is an IFN with membership and non-membership functions given by (1) and (2) respectively.

$$f_{\tilde{A}^I}(x) = \begin{cases} \frac{1}{n} \left[\frac{x-a_i}{a_{i+1}-a_i} \right] + \frac{i}{n} & ; a_i \leq x \leq a_{i+1}, \quad i = 0, \dots, n-1 \\ 1 & ; a_n \leq x \leq b_0 \\ \frac{-1}{n} \left[\frac{x-b_i}{b_{i+1}-b_i} \right] + \frac{n-i}{n} & ; b_i \leq x \leq b_{i+1}, \quad i = 0, \dots, n-1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and

$$g_{\tilde{A}^I}(x) = \begin{cases} \frac{-1}{n} \left[\frac{x-a'_i}{a'_{i+1}-a'_i} \right] + \frac{n-i}{n} & ; a'_i \leq x \leq a'_{i+1}, \quad i = 0, \dots, n-1 \\ 0 & ; a'_n \leq x \leq b'_0 \\ \frac{1}{n} \left[\frac{x-b'_i}{b'_{i+1}-b'_i} \right] + \frac{i}{n} & ; b'_i \leq x \leq b'_{i+1}, \quad i = 0, \dots, n-1 \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

The set of all n -IPFN's is denoted by \mathcal{IP}_n . An intuitionistic fuzzy number with $4n$ knots is represented as:

$$\tilde{A}^I = \{(a_0, a_1, \dots, a_n; b_0, b_1, \dots, b_n), (a'_0, a'_1, \dots, a'_n; b'_0, b'_1, \dots, b'_n)\}$$

An example of a 2-IPFN is given in Figure 1.

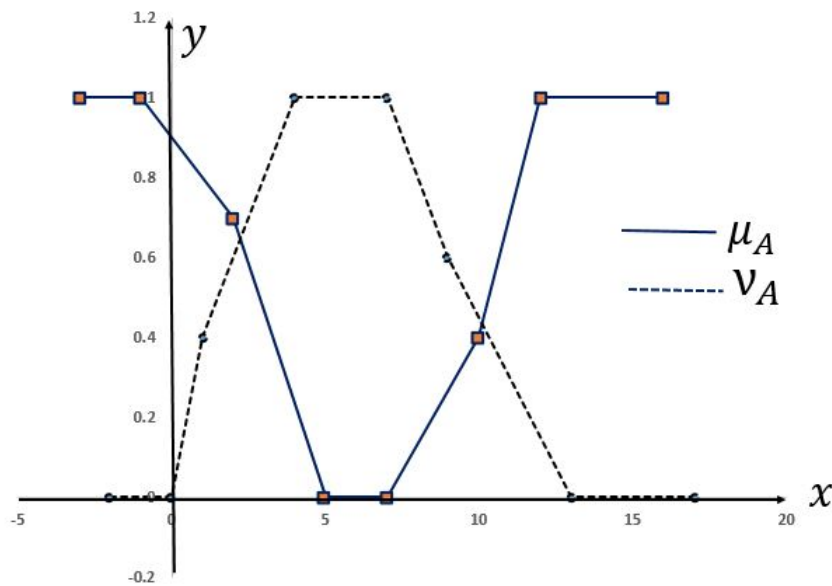


Figure 1. An example of a 2-IPFN

We use the following ranking function for IPFN.

Definition 4. [24] Let $\tilde{A}^I = \langle \mathbb{R}, \mu_{\tilde{A}^I}, \nu_{\tilde{A}^I} \rangle$ be an intuitionistic fuzzy number and let $\tilde{A}^I = \{(a_0, a_1, \dots, a_n; b_0, b_1, \dots, b_n), (a'_0, a'_1, \dots, a'_n; b'_0, b'_1, \dots, b'_n)\}$, then the ranking function is given by (3).

$$\mathfrak{R}(\tilde{A}^I) = \frac{1}{8n} [a_0 + 2a_1 + 2a_2 + \dots + 2a_{n-1} + a_n + b_0 + 2b_1 + 2b_2 + \dots + 2b_{n-1} + b_n + a'_0 + 2a'_1 + 2a'_2 + \dots + 2a'_{n-1} + a'_n + b'_0 + 2b'_1 + 2b'_2 + \dots + 2b'_{n-1} + b'_n] \quad (3)$$

Let \tilde{A}^I and $\tilde{B}^I \in \mathcal{IP}_n$ be given by (4) and (5).

$$\tilde{A}^I = \{(a_0, a_1, \dots, a_n; b_0, b_1, \dots, b_n), (c_0, c_1, \dots, c_n; d_0, d_1, \dots, d_n)\} \quad (4)$$

$$\tilde{B}^I = \{(a'_0, a'_1, \dots, a'_n; b'_0, b'_1, \dots, b'_n), (c'_0, c'_1, \dots, c'_n; d'_0, d'_1, \dots, d'_n)\} \quad (5)$$

then the relations between any two intuitionistic numbers is defined based on the ranking function as follows:

- \tilde{A}^I and \tilde{B}^I are called equivalent, denoted $\tilde{A}^I \approx \tilde{B}^I$, if $\mathfrak{R}(\tilde{A}^I) = \mathfrak{R}(\tilde{B}^I)$,
- they are called equal, denoted $\tilde{A}^I = \tilde{B}^I$, if $a_i = a'_i, b_i = b'_i, c_i = c'_i$ and $d_i = d'_i$ for all $i = 0, 1, \dots, n$,
- $\tilde{A}^I \succeq \tilde{B}^I$ if $\mathfrak{R}(\tilde{A}^I) \geq \mathfrak{R}(\tilde{B}^I)$.

Let \tilde{A}^I and $\tilde{B}^I \in \mathcal{IP}_n$ be given in (4) and (5), then arithmetic operations on the IPFN are given as follows;

Definition 5. [24] The addition of \tilde{A}^I and \tilde{B}^I is defined as follows:

$$\tilde{A}^I \oplus \tilde{B}^I = \{(a_0 + a'_0, a_1 + a'_1, \dots, a_n + a'_n; b_0 + b'_0, b_1 + b'_1, \dots, b_n + b'_n), (c_0 + c'_0, c_1 + c'_1, \dots, c_n + c'_n; d_0 + d'_0, d_1 + d'_1, \dots, d_n + d'_n)\}$$

The multiplication of intuitionistic number by a positive crisp number is to multiply each node by this number $\tilde{k}A^I = \{(ka_0, ka_1, \dots, ka_n; kb_0, kb_1, \dots, kb_n), (kc_0, kc_1, \dots, kc_n; kd_0, kd_1, \dots, kd_n)\}$. However, if k is negative then $\tilde{k}A^I = \{(ka_n, ka_{n-1}, \dots, ka_1; kb_n, kb_{n-1}, \dots, kb_0), (kc_n, kc_{n-1}, \dots, kc_0; kd_n, kd_{n-1}, \dots, kd_0)\}$. The subtraction of intuitionistic numbers $\tilde{A}^I \ominus \tilde{B}^I$ is given by

$$\tilde{A}^I \ominus \tilde{B}^I = \{(a_0 - a'_0, a_1 - a'_{n-1}, \dots, a_n - a'_0; b_0 - b'_n, b_1 - b'_{n-1}, \dots, b_n - b'_0), (c_0 - c'_n, c_1 - c'_{n-1}, \dots, c_n - c'_0; d_0 - d'_n, d_1 - d'_{n-1}, \dots, d_n - d'_0)\}$$

Definition 6. [24] The product $\tilde{A}^I \otimes \tilde{B}^I$ is defined as follows, assume that

$$\tilde{A}^I \otimes \tilde{B}^I = \{(e_0, e_1, \dots, e_n; f_0, f_1, \dots, f_n), (g_0, g_1, \dots, g_n; h_0, h_1, \dots, h_n)\},$$

The values of are the solution of (6) – (8):

$$e_0 + 2e_1 + 2e_2 + \dots + 2e_{n-1} + e_n + f_0 + 2f_1 + 2f_2 + \dots + 2f_{n-1} + f_n + g_0 + 2g_1 + 2g_2 + \dots + 2g_{n-1} + g_n + h_0 + 2h_1 + 2h_2 + \dots + 2h_{n-1} + h_n = I \quad (6)$$

where

$$I = \frac{1}{8n} [(a_0 + 2a_1 + 2a_2 + \dots + 2a_{n-1} + a_n + b_0 + 2b_1 + 2b_2 + \dots + 2b_{n-1} + b_n + c_0 + 2c_1 + 2c_2 + \dots + 2c_{n-1} + c_n + c_0 + 2c_1 + 2c_2 + \dots + 2c_{n-1} + c_n) \cdot (a'_0 + 2a'_1 + 2a'_2 + \dots + 2a'_{n-1} + a'_n + b'_0 + 2b'_1 + 2b'_2 + \dots + 2b'_{n-1} + b'_n + c'_0 + 2c'_1 + 2c'_2 + \dots + 2c'_{n-1} + c'_n + c'_0 + 2c'_1 + 2c'_2 + \dots + 2c'_{n-1} + c'_n)]. \quad (7)$$

For $i = 1, 2, \dots, n$ we let:

$$\begin{aligned}
 e_i - e_{i-1} &= (a_i - a_{i-1}) + (a'_i - a'_{i-1}), \\
 f_0 - e_n &= (b_0 - a_n) + (b'_0 - a'_n), \\
 f_i - f_{i-1} &= (b_i - b_{i-1}) + (b'_i - b'_{i-1}), \\
 g_i - g_{i-1} &= (c_i - c_{i-1}) + (c'_i - c'_{i-1}), \\
 h_0 - g_n &= (d_0 - c_n) + (d'_0 - c'_n), \\
 h_i - h_{i-1} &= (d_i - d_{i-1}) + (d'_i - d'_{i-1}), \\
 f_0 - h_0 &= b_0 + b'_0 - d_0 - d'_0
 \end{aligned} \tag{8}$$

Note that (6) and (7) guarantee that the ranking of the product of two intuitionistic numbers is the product of the ranking of the intuitionistic numbers: $\mathfrak{R}(\tilde{A}^I \otimes \tilde{B}^I) = \mathfrak{R}(\tilde{A}^I)\mathfrak{R}(\tilde{B}^I)$.

3. THE INTUITIONISTIC FULLY FUZZY LINEAR PROGRAMMING (IFFLP) WITH N-IPFN

The standard form of a IFFLP problem with n -IPFN's is given by (9):

$$\begin{aligned}
 \min \quad & \tilde{z}^I = \tilde{c}^I \tilde{x}^I \\
 \text{s.t.} \quad & \tilde{A}^I \tilde{x}^I \approx \tilde{b}^I \\
 & \tilde{x}^I \succeq \tilde{0}^I
 \end{aligned} \tag{9}$$

where $\tilde{c}^I = [\tilde{c}_j^I]_{1 \times l}$, $\tilde{x}^I = [\tilde{x}_j^I]_{l \times 1}$, $\tilde{A}^I = [\tilde{a}_{ij}^I]_{m \times l}$, $\tilde{b}^I = [\tilde{b}_i^I]_{m \times 1}$ are matrices with n -IPFN's entries. Moreover, $\tilde{b}_i^I \succeq \tilde{0}^I$ for all $i = 1, \dots, m$, and the matrix \tilde{A}^I is of rank m .

An IFFLP problem may not be in standard form because of some inequality constraints, unrestricted variables or having an objective function to be maximized. In such cases, we can transform the problem into the standard form as follows:

- An inequality of the form “ $\tilde{a}_i^I \tilde{x}^I \preceq \tilde{b}_i^I$ ” can be transformed into an equality constraint by adding a non-negative fuzzy unknown value \tilde{s}^I , called a *slack intuitionistic fuzzy variable*, such that the constraint becomes $\tilde{a}_i^I \tilde{x}^I \oplus \tilde{s}^I \approx \tilde{b}_i^I$.
- If the inequality constraint is in the form “ $\tilde{a}_i^I \tilde{x}^I \succeq \tilde{b}_i^I$ ”, then we may subtract a non-negative *surplus intuitionistic fuzzy variable* \tilde{p}^I to transform the constraint into $\tilde{a}_i^I \tilde{x}^I \ominus \tilde{p}^I \approx \tilde{b}_i^I$.
- An unrestricted variable \tilde{x}_j^I can be replaced by two nonnegative variables, \tilde{x}_j^{I+} and \tilde{x}_j^{I-} , by putting $\tilde{x}_j^I = \tilde{x}_j^{I+} \oplus \tilde{x}_j^{I-}$.
- Finally, if the objective function is to be maximized, then taking the additive inverse of the objective value with the same values for the variables gives the solution of the original maximization problem.

3.1. Basic feasible solutions

After possibly rearranging the columns \tilde{a}_j^I of \tilde{A}^I , let $\tilde{A}^I = [\tilde{B}^I \quad \tilde{N}^I]$, where \tilde{B}^I is an $m \times m$ invertible matrix consisting of m columns of \tilde{a}_j^I , and \tilde{N}^I is an $m \times (l - m)$ matrix with the rest of the columns. Then the constraints can be written as:

$$\begin{aligned}
 \tilde{A}^I \tilde{x}^I &\approx \tilde{b}^I \\
 [\tilde{B}^I \quad \tilde{N}^I] \tilde{x}^I &\approx \tilde{b}^I
 \end{aligned}$$

The variables vector can then be split as (10):

$$\begin{aligned}
 [\tilde{B}^I \quad \tilde{N}^I] \begin{bmatrix} \tilde{x}_B^I \\ \tilde{x}_N^I \end{bmatrix} &\approx \tilde{b}^I \\
 \tilde{B}^I \tilde{x}_B^I \oplus \tilde{N}^I \tilde{x}_N^I &\approx \tilde{b}^I \\
 \tilde{x}_B^I \oplus \tilde{B}^{I(-1)} \tilde{N}^I \tilde{x}_N^I &\approx \tilde{B}^{I(-1)} \tilde{b}^I
 \end{aligned} \tag{10}$$

One solution is $\tilde{x}^I = \begin{bmatrix} \tilde{x}_B^I \\ \tilde{x}_N^I \end{bmatrix} = \begin{bmatrix} \tilde{B}^{I(-1)} \tilde{b}^I \\ \tilde{0}^I \end{bmatrix}$, which is called a *basic solution*. \tilde{B}^I is called the *basis*, and the components of \tilde{x}_B^I are called the *basic variables*. If $\tilde{x}_B^I \succeq \tilde{0}$, then \tilde{x}^I is called a *basic feasible solution (b.f.s.)*.

4. THE INTUITIONISTIC FUZZY SIMPLEX METHOD

Assume problem (9) has a basic feasible solution $\tilde{x}^I = \begin{bmatrix} \tilde{x}_B^I \\ \tilde{x}_N^I \end{bmatrix} = \begin{bmatrix} \tilde{B}^{I(-1)} \tilde{b}^I \\ \tilde{0}^I \end{bmatrix}$, whose objective value is given by:

$$\tilde{z}_0^I = \tilde{c}^I \begin{bmatrix} \tilde{B}^{I(-1)} \tilde{b}^I \\ \tilde{0}^I \end{bmatrix} = [\tilde{c}_B^I \quad \tilde{c}_N^I] \begin{bmatrix} \tilde{B}^{I(-1)} \tilde{b}^I \\ \tilde{0}^I \end{bmatrix} = \tilde{c}_B^I \tilde{B}^{I(-1)} \tilde{b}^I \tag{11}$$

The objective function in augmented form is:

$$\tilde{z}^I = [\tilde{c}_B^I \quad \tilde{c}_N^I] \begin{bmatrix} \tilde{x}_B^I \\ \tilde{x}_N^I \end{bmatrix} = \tilde{c}_B^I \tilde{x}_B^I \oplus \tilde{c}_N^I \tilde{x}_N^I \tag{12}$$

From (10), we have (13):

$$\tilde{x}_B^I \approx \tilde{B}^{I(-1)} \tilde{b}^I \ominus \tilde{B}^{I(-1)} \tilde{N}^I \tilde{x}_N^I \tag{13}$$

Substituting (13) in (12) and simplifying:

$$\tilde{z}^I \oplus (\tilde{c}_B^I \tilde{B}^{I(-1)} \tilde{N}^I \ominus \tilde{c}_N^I) \tilde{x}_N^I \approx \tilde{c}_B^I \tilde{B}^{I(-1)} \tilde{b}^I$$

Let $\tilde{z}_N^I = \tilde{c}_B^I \tilde{B}^{I(-1)} \tilde{N}^I$, then:

$$\tilde{z}^I \oplus (\tilde{z}_N^I \ominus \tilde{c}_N^I) \tilde{x}_N^I \approx \tilde{c}_B^I \tilde{B}^{I(-1)} \tilde{b}^I \tag{14}$$

From (10) and (14), and putting $\tilde{b}^I = \tilde{B}^{I(-1)} \tilde{b}^I$, the current b.f.s. can be represented in the tableau form as in Table 1:

Table 1. A tableau form for the simplex method

| | \tilde{x}_B^I | \tilde{x}_N^I | RHS |
|-----------------|-----------------|---------------------------------------|---|
| \tilde{z}^I | $\tilde{0}^I$ | $\tilde{z}_N^I \ominus \tilde{c}_N^I$ | $\tilde{c}_B^I \tilde{B}^{I(-1)} \tilde{b}^I$ |
| \tilde{x}_B^I | \tilde{I}^I | $\tilde{B}^{I(-1)} \tilde{N}^I$ | \tilde{b}^I |

Note that Row 0 (the second row in the table) represents the objective value of the current solution $\tilde{z}^I = \tilde{c}_B^I \tilde{B}^{I(-1)} \tilde{b}^I$ since $\tilde{x}_N^I = 0$. Row 1 (the third row in the table is in fact m rows represent the values of the basic variables $\tilde{x}_B^I = \tilde{b}^I$.)

Without loss of generality, we assume the absence of *degeneracy*, i.e. we assume that $\tilde{b}^I \succ 0$. The case of degeneracy, where \tilde{b}^I has zero values is known to cause some problems and needs a special discussion that will be studied later.

Let J be the current set of indices of the non-basic variables, then $\tilde{z}_j^I \ominus \tilde{c}_j^I$ where $j \in J$ are the elements of $\tilde{z}_N^I \ominus \tilde{c}_N^I$. Now, from (14) we have:

$$\tilde{z}^I \approx \tilde{z}_0^I \oplus \sum_{j \in J} (\tilde{z}_j^I \ominus \tilde{c}_j^I) \tilde{x}_j^I \tag{15}$$

Note that if $\tilde{z}_j^I \ominus \tilde{c}_j^I < 0$ for all $j \in J$, then increasing the value of any nonbasic variable of the \tilde{x}_j^I 's increases the value of the objective function \tilde{z}^I . Therefore, the current solution cannot be improved anymore, and it is optimal. On the other hand, if $\tilde{z}_j^I \ominus \tilde{c}_j^I \leq 0$ for all $j \in J$, and $\tilde{z}_k^I \ominus \tilde{c}_k^I \approx 0$ for some $k \in J$, then increasing the value of \tilde{x}_k^I does not affect the objective value, which means that we have alternative optimal solutions with the

same objective value. However, such case is not treated differently than the previous case in this paper. In other words, even if we have alternative optimal solutions, we will take only one of them into consideration. Finally, if there exists $\tilde{z}_k^I \ominus \tilde{c}_k^I \succ 0$ for some $k \in J$, then increasing the ranking value of \tilde{x}_j^I obviously decreases the objective value, which means that the current solution is not optimal and there is a better one. Thus, \tilde{x}_k^I needs to enter the basis, and another variable needs to leave it.

Determining the leaving variable can be done as follows:

Let \tilde{y}_j^I for $j \in J$ be the columns of the matrix $\tilde{B}^{I(-1)}\tilde{N}^I$, i.e. $\tilde{y}_j^I = \tilde{B}^{I(-1)}\tilde{a}_j^I = [\tilde{y}_{ij}^I]$. Since the nonbasic variables other than \tilde{x}_k^I will stay nonbasic with zero value, then from (13) and (15) the problem can be written as:

$$\begin{aligned} \min \quad & \tilde{z}^I \approx \tilde{z}_0^I \ominus (\tilde{z}_k^I \ominus \tilde{c}_k^I)\tilde{x}_k^I \\ \text{s.t.} \quad & \begin{bmatrix} \tilde{x}_{B_1}^I \\ \tilde{x}_{B_2}^I \\ \dots \\ \tilde{x}_{B_m}^I \end{bmatrix} \approx \begin{bmatrix} \tilde{b}_1^I \\ \tilde{b}_2^I \\ \dots \\ \tilde{b}_m^I \end{bmatrix} \ominus \begin{bmatrix} \tilde{y}_{1k}^I \\ \tilde{y}_{2k}^I \\ \dots \\ \tilde{y}_{mk}^I \end{bmatrix} \otimes \tilde{x}_k^I \\ & \tilde{x}_k^I \succeq 0, \quad \tilde{x}_{B_i}^I \succeq 0 \quad \forall i = 1, \dots, m \end{aligned}$$

where $\tilde{x}_{B_i}^I$ are the elements of \tilde{x}_B^I (i.e. the basic variables). The constraints can be written as:

$$\tilde{x}_{B_i}^I \approx \tilde{b}_i^I \ominus \tilde{y}_{ik}^I \otimes \tilde{x}_k^I \quad \forall i = 1, \dots, m$$

For each i , we note that:

- If $\tilde{y}_{ik}^I \leq 0$, then $\tilde{x}_{B_i}^I$ increases or does not get affected as \tilde{x}_k^I increases, so $\tilde{x}_{B_i}^I$ continues to be nonnegative.
- If $\tilde{y}_{ik}^I \succ 0$, then $\tilde{x}_{B_i}^I$ will decrease as \tilde{x}_k^I increases.

In order to maintain the nonnegativity of the variables, \tilde{x}_k^I is increased until the first point at which some basic variable $\tilde{x}_{B_r}^I$ drops to zero. In fact, we can increase \tilde{x}_k^I until:

$$\tilde{x}_k^I = \frac{\tilde{b}_r^I}{\tilde{y}_{rk}^I} = \min \left\{ \frac{\tilde{b}_i^I}{\tilde{y}_{ik}^I}; \tilde{y}_{ik}^I \succ 0, i = 1, \dots, m \right\} \quad (16)$$

and then, $\tilde{x}_{B_r}^I$ is the variable that leaves the basis and we call it the *blocking variable*, and (16) is called the *minimum ratio*. In fact, the only purpose of finding the minimum ratio (16) is to determine the blocking variable. Therefore, we can use the ranking function to facilitate the calculations, and the following *ranked minimum ratio* is enough to achieve the purpose:

$$\frac{\Re[\tilde{b}_r^I]}{\Re[\tilde{y}_{rk}^I]} = \min \left\{ \frac{\Re[\tilde{b}_i^I]}{\Re[\tilde{y}_{ik}^I]}; \tilde{y}_{ik}^I \succ 0, i = 1, \dots, m \right\} \quad (17)$$

In the tableau format, we can change the basis using the elementary row operations, which are known to maintain an equivalent problem, such that \tilde{x}_k^I enters the basis and $\tilde{x}_{B_r}^I$ leaves it. One possible case still needs to be discussed, that is when $\tilde{y}_k^I \leq 0$, i.e. the ranking values of all its elements are less than or equivalent to zero. In this case, there is no blocking variable, and the value of \tilde{x}_k^I can be increased indefinitely giving always a better objective value without violating any of the constraints. Thus, the problem is unbounded.

Trying to update the solution gives:

$$\begin{aligned} \tilde{x}_B^I &= \tilde{b}^I \ominus (\tilde{y}_k^I \otimes \tilde{x}_k^I) \\ \tilde{x}_N^I &= \tilde{e}_k^I \otimes \tilde{x}_k^I \end{aligned}$$

where \tilde{e}_k^I is a $(l - m)$ -vector with zero entries, except 1 in the k -th position. Then the new solution becomes:

$$\begin{bmatrix} \tilde{x}_B^I \\ \tilde{x}_N^I \end{bmatrix} \approx \begin{bmatrix} \tilde{B}^{I(-1)}\tilde{b}^I \\ \tilde{0}^I \end{bmatrix} \oplus \begin{bmatrix} -\tilde{y}_k^I \\ \tilde{e}_k^I \end{bmatrix} \otimes \tilde{x}_k^I$$

We call the vector $\tilde{d}^I = \begin{bmatrix} -\tilde{y}_k^I \\ \tilde{e}_k^I \end{bmatrix}$ the direction of unboundedness, which satisfies:

- $\tilde{d}^I \succeq 0$ clearly.
- $\tilde{c}^I \tilde{d}^I \prec 0$ since: $\tilde{c}^I \tilde{d}^I = [\tilde{c}_B^I \ \tilde{c}_N^I] \tilde{d}^I = -\tilde{c}_B^I \tilde{y}_k^I \oplus \tilde{c}_k^I = -\tilde{z}_k^I \oplus \tilde{c}_k^I \prec 0$
- $\tilde{A}^I \tilde{d}^I \approx 0$ since:

$$\begin{aligned} \tilde{A}^I \otimes \tilde{d}^I &= [\tilde{B}^I \ \tilde{N}^I] \otimes \begin{bmatrix} -\tilde{y}_k^I \\ \tilde{c}_k^I \end{bmatrix} = (\tilde{B}^I \otimes (-\tilde{y}_k^I)) \oplus (\tilde{N}^I \otimes \tilde{c}_k^I) \\ &= (\tilde{B}^I \otimes (-\tilde{B}^{I(-1)} \otimes \tilde{a}_k^I)) \oplus \tilde{a}_k^I \approx (-\tilde{I}^I \otimes \tilde{a}_k^I) \oplus \tilde{a}_k^I \approx -\tilde{a}_k^I \oplus \tilde{a}_k^I \approx 0 \end{aligned}$$

In fact, having a vector \tilde{d}^I satisfying these three properties is a necessary and sufficient condition for the unboundedness of the problem. To show this, let \tilde{d}^I be a l -vector satisfying: $\tilde{A}^I \tilde{d}^I \approx 0$, $\tilde{c}^I \tilde{d}^I \prec 0$ and $\tilde{d}^I \succeq 0$, and suppose that the problem is bounded, i.e. it has an optimal solution \tilde{x}^{I*} with the optimal objective value $\tilde{z}^{I*} = \tilde{c}^I \tilde{x}^{I*}$. Now let $\tilde{x}^{I'} = \tilde{x}^{I*} \oplus \tilde{d}^I$ this maintains the nonnegativity of the variables since $\tilde{d}^I \succeq 0$ and it is a feasible solution since:

$$\tilde{A}^I \tilde{x}^{I'} = \tilde{A}^I \otimes (\tilde{x}^{I*} \oplus \tilde{d}^I) = \tilde{A}^I \tilde{x}^{I*} \oplus \tilde{A}^I \tilde{d}^I \approx \tilde{A}^I \tilde{x}^{I*} \approx \tilde{b}^I$$

since \tilde{x}^{I*} is feasible. On the other hand, the objective value at $\tilde{x}^{I'}$ is:

$$\tilde{z}^{I'} = \tilde{c}^I \otimes (\tilde{x}^{I*} \oplus \tilde{d}^I) = \tilde{c}^I \tilde{x}^{I*} \oplus \tilde{c}^I \tilde{d}^I \prec \tilde{c}^I \tilde{x}^{I*} = \tilde{z}^{I*}$$

which means the $\tilde{x}^{I'}$ gives a better objective value and is a better solution, which contradicts with the assumption that \tilde{x}^{I*} is the optimal solution.

4.1. Summarizing the modified simplex algorithm

To summarize, assuming a minimization problem, the simplex algorithm proceeds as follows:

Step 1: Let $\tilde{z}_k^I \ominus \tilde{c}_k^I = \max\{\tilde{z}_j^I \ominus \tilde{c}_j^I : j \in J\}$. If $\tilde{z}_k^I \ominus \tilde{c}_k^I \preceq \tilde{0}^I$, then stop; the current solution is optimal. Otherwise, proceed to the next step.

Step 2: If $\tilde{y}_k^I \preceq \tilde{0}^I$, then stop; the solution is unbounded. Otherwise, proceed.

Step 3: Let $\frac{\Re[\tilde{b}_r^I]}{\Re[\tilde{y}_{rk}^I]} = \min\{\frac{\Re[\tilde{b}_i^I]}{\Re[\tilde{y}_{ik}^I]}; \tilde{y}_{ik}^I \succ 0, i = 1, \dots, m\}$, then \tilde{x}_k^I enters the basis and $\tilde{x}_{B_r}^I$ leaves it, so pivot at \tilde{y}_{rk}^I as follows:

- Multiply row r by the multiplicative inverse of \tilde{y}_{rk}^I .
- Update the other rows using the following elementary row operations: $R'_0 = (-\tilde{z}_k^I \ominus \tilde{c}_k^I) \otimes R_r \oplus R_0$, $R'_i = (-\tilde{y}_{ik}^I \otimes R_r) \oplus R_i, i \in \{1, \dots, m\} \setminus \{r\}$, where R_0 is row zero (i.e. the objective row), and R_i is row number i in the tableau after the objective row.

Then, update the set of indices of the non-basic variables J , and go to Step 1.

Remark 1. Note that the choice of the pivoting location depends on the ranking values, and the pivoting steps consist of elementary row operations which depend on the arithmetic operations on the n -IPFN that preserve the ranking values. This results in the following important property: Suppose that we replace each intuitionistic fuzzy number in problem (9) by its ranking value, and replace the intuitionistic fuzzy variables with crisp (unfuzzy) ones. Then we get the crisp linear programming problem:

$$\begin{aligned} \min \quad & z = cx \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{18}$$

We call this problem the ranked linear programming (RLP) problem. It is clear that the values of the variables in the optimal solution of problem (18) equal the ranking values of the fuzzy variables in the optimal solution of problem (9). Moreover, the steps of solving the original problem are equivalent to the steps of solving the corresponding RLP problem in number and order, which means that the fuzzy simplex method proposed in this paper terminates in a finite number of iterations.

5. NUMERICAL EXAMPLES

The following example illustrates the above.

Example 1. Consider the following IFFLP problem:

$$\begin{aligned}
 \min & \{(1, 2; 4, 5), (-1, 2; 5, 6)\} \otimes \tilde{x}_1^I \oplus \{(-6, -2.5; -1, 1), (-7.5, -4; 1, 3)\} \otimes \tilde{x}_2^I \\
 s.t. & \{(-4, -2; -0.5, 2), (-5, -3.5; 1, 4)\} \otimes \tilde{x}_1^I \oplus \{(2, 3; 3, 5), (0, 2; 3, 6)\} \otimes \tilde{x}_2^I \preceq \{(4, 5; 6, 9), (2, 4; 8, 10)\} \\
 & \{(1, 3; 5, 6), (0, 3; 6, 8)\} \otimes \tilde{x}_1^I \oplus \{(-1, 1; 3, 6), (-3, 0; 4, 6)\} \otimes \tilde{x}_2^I \preceq \{(4, 6; 9, 10), (1, 5; 10, 11)\} \\
 & \tilde{x}_1^I, \tilde{x}_2^I \succeq \tilde{0}^I
 \end{aligned} \tag{19}$$

Adding the slack variables, Table 2 gives the first simplex tableau:

Table 2. The first tableau for Example 1

| | \tilde{x}_1^I | \tilde{x}_2^I | \tilde{y}_1^I | \tilde{y}_2^I | RHS |
|-----------------|---|---|-----------------|-----------------|-------------------------------------|
| \tilde{z}^I | $\{(-5, -4; -2, -1), (-6, -5; -2, 1)\}$ | $\{(-1, 1; 2.5, 6), (-3, -1; 4, 7.5)\}$ | 0 | 0 | 0 |
| RV | -3 | 2 | | | |
| \tilde{y}_1^I | $\{(-4, -2; -0.5, 2), (-5, -3.5; 1, 4)\}$ | $\{(2, 3; 3, 5), (0, 2; 3, 6)\}$ | 1 | 0 | $\{(4, 5; 6, 9), (2, 4; 8, 10)\}$ |
| RV | -1 | $\triangleright 3 \triangleleft$ | | | 6 |
| \tilde{y}_2^I | $\{(1, 3; 5, 6), (0, 3; 6, 8)\}$ | $\{(-1, 1; 3, 6), (-3, 0; 4, 6)\}$ | 0 | 1 | $\{(4, 6; 9, 10), (1, 5; 10, 11)\}$ |
| RV | 4 | 2 | | | 7 |

where the ranking value (RV) of each intuitionistic fuzzy number is written below it.

$$\begin{aligned}
 \tilde{z}_k^I \ominus \tilde{c}_k^I &= \max\{\{(-5, -4; -2, -1), (-6, -5; -2, 1)\}, \{(-1, 1; 2.5, 6), (-3, -1; 4, 7.5)\}, 0, 0\} \\
 &= \{(-1, 1; 2.5, 6), (-3, -1; 4, 7.5)\} = \tilde{z}_2^I \ominus \tilde{c}_2^I \succ \tilde{0},
 \end{aligned}$$

thus the current solution is not optimal and the non basic variable \tilde{z}_2^I enters the basis. Using the ranked minimum ratio test (17), we find:

$$\frac{\mathfrak{R}[\tilde{b}_r^I]}{\mathfrak{R}[\tilde{y}_{rk}^I]} = \min\left\{\frac{6}{3}, \frac{7}{2}\right\} = \frac{\mathfrak{R}[\tilde{b}_1^I]}{\mathfrak{R}[\tilde{y}_{12}^I]} = 2$$

Therefore, \tilde{y}_1^I leaves the basis and we pivot at $\{(2, 3; 3, 5), (0, 2; 3, 6)\}$ by performing the elementary row operations:

$$\begin{aligned}
 R_1 &\leftarrow \{(2, 3; 3, 5), (0, 2; 3, 6)\}^{-1} \otimes R_1 \\
 R_0 &\leftarrow -\{(-1, 1; 2.5, 6), (-3, -1; 4, 7.5)\} \otimes R_1 \oplus R_0 \\
 R_2 &\leftarrow -\{(-1, 1; 3, 6), (-3, 0; 4, 6)\} \otimes R_1 \oplus R_2
 \end{aligned}$$

This gives the second simplex tableau in Table 3:

Table 3. The final tableau for Example 1

| | \tilde{x}_1 | \tilde{x}_2 | \tilde{y}_1 | \tilde{y}_2 | RHS |
|---------------|--|--|---|--|--|
| \tilde{z} | $\{(-\frac{40}{3}, -\frac{29}{6}; \frac{1}{6}, \frac{20}{3}), (-\frac{10}{6}, -\frac{5}{6}; \frac{14}{3}, \frac{44}{3})\}$ | $\{(-10, -\frac{3}{2}; \frac{3}{2}, 10), (-\frac{33}{2}, -6; 6, \frac{33}{2})\}$ | $\{(-\frac{20}{3}, -\frac{7}{6}; \frac{1}{3}, \frac{10}{3}), (-\frac{55}{6}, -\frac{8}{3}; \frac{10}{3}, \frac{22}{3})\}$ | $\{(-6, -0.5; 1, 4), (-8.5, -2; 4, 8)\}$ | $\{(-12, -\frac{11}{2}; -3, 3), (-\frac{33}{2}; -8, 2, 8)\}$ |
| RV | $-\frac{7}{3}$ | 0 | $-\frac{2}{3}$ | 0 | -4 |
| \tilde{x}_2 | $\{(-\frac{16}{3}, -\frac{4}{3}; \frac{1}{3}, \frac{11}{3}), (-\frac{22}{3}, -\frac{1}{6}; \frac{1}{6}, \frac{23}{3})\}$ | $\{(-2, 1; 1, 4), (-5, 0; 2, 7)\}$ | $\{(-\frac{5}{3}, \frac{1}{3}; \frac{1}{3}, \frac{4}{3}), (-\frac{8}{3}, \frac{1}{3}; \frac{4}{3}, \frac{10}{3})\}$ | $\{(-2, 0; 0, 1), (-3, 0; 1, 3)\}$ | $\{(-2, 1; 2, 6), (-5, 0; 5, 9)\}$ |
| RV | $-\frac{1}{3}$ | 1 | $-\frac{1}{3}$ | 0 | 2 |
| \tilde{y}_2 | $\{(-\frac{22}{3}, \frac{5}{3}; \frac{43}{6}, \frac{41}{3}), (-\frac{31}{3}, -\frac{8}{6}; \frac{32}{3}, \frac{65}{3})\}$ | $\{(-10, -2; 2, 10), (-15, -5; 5, 15)\}$ | $\{(-\frac{20}{3}, -\frac{5}{3}; \frac{1}{3}, \frac{10}{3}), (-\frac{23}{3}, -\frac{8}{3}; \frac{2}{3}, \frac{22}{3})\}$ | $\{(-5, 0; 2, 5), (-6, -1; 4, 9)\}$ | $\{(-8, 0; 6, 13), (-14, -3; 11, 19)\}$ |
| RV | $\frac{14}{3}$ | 0 | $-\frac{20}{3}$ | 1 | 3 |

$\tilde{z}_k \ominus \tilde{c}_k = 0$, thus the solution is optimal. The optimal solution for the problem is: $\tilde{x}_1^* = 0, \tilde{x}_2^* = \{(-2, 1; 2, 6), (-5, 0; 5, 9)\}$ with the fuzzy objective value $\tilde{z}^* = \{(-12, -\frac{11}{2}; -3, 3), (-\frac{33}{2}; -8, 2, 8)\}$. Now, if we solve the RLP problem for problem (19), which is:

$$\begin{aligned} \min \quad & 3x_1 - 2x_2 \\ \text{s.t.} \quad & -x_1 + 3x_2 \leq 6 \\ & 4x_1 + 2x_2 \leq 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

we find that the optimal solution is: $x_1^* = 0, x_2^* = 2$ with the optimal objective value $z^* = -4$. As expected, we have $x_1^* = \mathfrak{R}(\tilde{x}_1^*), x_2^* = \mathfrak{R}(\tilde{x}_2^*)$ and $z^* = \mathfrak{R}(\tilde{z}^*)$.

The next example is an unbounded IFFLP with 3-IPFN's.

Example 2. Consider the following problem:

$$\begin{aligned} \max \quad & \{(-3, -1, 1, 2; 2, 3, 5, 7), (-4, -1.5, 0.5, 2; 3, 3.5, 5, 8)\} \otimes \tilde{x}_1^I \oplus \\ & \{(-10, -7, -6, -4; -2, 0, 1, 4), (-12, -9, -6, -4.5; -1, 1, 2, 5.5)\} \otimes \tilde{x}_2^I \\ \text{s.t.} \quad & \{(-5, -4, -3, -2; -1, 2, 2.5, 3), (-8, -5, -3, -2; 0, 1, 3, 4)\} \otimes \tilde{x}_1^I \oplus \\ & \{(-4, -3, -1, 0; 2, 3, 5, 6), (-7, -5, -2, 0; 3, 5, 6, 8)\} \otimes \tilde{x}_2^I \preceq \\ & \{(-2, 0, 1, 2; 2, 3, 4, 6), (-6, -2, -1, 1; 3, 5, 7, 8)\} \\ & \{(-4, -3, -2, -1; -1, 0, 1, 2), (-10, -5, -3, -2; 1, 2, 3, 5)\} \otimes \tilde{x}_1^I \oplus \\ & \{(-3, -1, 1, 2; 2, 3, 5, 7), (-6, -3, -2, 0; 4, 5, 7, 12)\} \otimes \tilde{x}_2^I \succeq \\ & \{(-6, -5, -4, -2; -2, 0, 1, 2), (-18, -13, -8, -5; -1, 5, 9, 14)\} \\ & \tilde{x}_1^I, \tilde{x}_2^I \succeq \tilde{0}^I \end{aligned} \tag{20}$$

Note that this is a maximization problem, so we multiply row 1 by -1 to convert to a minimization problem and when we get an optimal solution, we multiply the objective value by -1 to get the actual optimal value for the maximization problem. Note also that the second constraint is a \succeq type inequality, so we multiply by -1 . Now, we add slack variables for both constraints to get the the first simplex tableau in Table 4. Since $\tilde{z}_k^I \ominus \tilde{c}_k^I = (-3, -1, 1, 2; 2, 3, 5, 7), (-4, -1.5, 0.5, 2; 3, 3.5, 5, 8)\} = \tilde{z}_1^I \ominus \tilde{c}_1^I \succ \tilde{0}$, the optimal solution is not reached. Using the minimum ratio test, the next iteration is by pivoting at $\{(-2, -1, 0, 1; 1, 2, 3, 4), (-5, -3, -2, -1; 2, 3, 5, 10)\}$ in the last row, which gives the tableau in Table 5. It is clear that the variable \tilde{x}_2 tries to enter the basis. However,

$$\tilde{y}_2 = \begin{bmatrix} \tilde{y}_{32} \\ \tilde{y}_{12} \end{bmatrix} = \begin{bmatrix} \{(-11, -8, -4, -2; 0, 2, 6, 9), (-19, -12, -7, -4; 3, 7, 9, 14)\} \\ \{(-7, -5, -3, -2; -2, -1, 1, 3), (-12, -7, -5, -4; 0, 2, 3, 6)\} \end{bmatrix} \preceq \tilde{0}$$

Therefore, the problem is unbounded with the direction of unboundedness:

$$\tilde{d} = \begin{bmatrix} -\tilde{y}_{12} \\ 1 \\ -\tilde{y}_{32} \\ 0 \end{bmatrix} = \begin{bmatrix} \{(-3, -1, 1, 2; 2, 3, 5, 7), (-6, -3, -2, 0; 4, 5, 7, 12)\} \\ \{(1, 1, 1, 1; 1, 1, 1, 1), (1, 1, 1, 1; 1, 1, 1, 1)\} \\ \{(-9, -6, -2, 0, 2, 4, 8, 11), (-14, -9, -7, -3; 4, 7, 12, 19)\} \\ \{(0, 0, 0, 0; 0, 0, 0, 0), (0, 0, 0, 0; 0, 0, 0, 0)\} \end{bmatrix}$$

Table 4. The first tableau for Example 2

| | \tilde{x}_1 | \tilde{x}_2 | \tilde{x}_3 | \tilde{x}_4 | RHS |
|---------------|---|--|---------------|---------------|--|
| \tilde{z} | $\{(-3, -1, 1, 2; 2, 3, 5, 7), (-4, -1.5, 0.5, 2; 3, 3.5, 5, 8)\}$ | $\{(-10, -7, -6, -4; -2, 0, 1, 4), (-12, -9, -6, -4.5; -1, 1, 2, 5.5)\}$ | 0 | 0 | 0 |
| RV | 2 | -3 | | | |
| \tilde{x}_3 | $\{(-5, -4, -3, -2; -1, 2, 2.5, 3), (-8, -5, -3, -2; 0, 1, 3, 4)\}$ | $\{(-4, -3, -1, 0; 2, 3, 5, 6), (-7, -5, -2, 0; 3, 5, 6, 8)\}$ | 1 | 0 | $\{(-2, 0, 1, 2; 2, 3, 4, 6), (-6, -2, -1, 1; 3, 5, 7, 8)\}$ |
| RV | -1 | 1 | | | 2 |
| \tilde{x}_4 | $\{(-2, -1, 0, 1; 1, 2, 3, 4), (-5, -3, -2, -1; 2, 3, 5, 10)\}$ | $\{(-7, -5, -3, -2; -2, -1, 1, 3), (-12, -7, -5, -4; 0, 2, 3, 6)\}$ | 0 | 1 | $\{(-2, -1, 0, 2; 2, 4, 5, 6), (-14, -9, -5, 1; 5, 8, 13, 18)\}$ |
| RV | $\triangleright 1 \triangleleft$ | -2 | | | 2 |

Table 5. The final tableau for Example 2

| | \tilde{x}_1 | \tilde{x}_2 | \tilde{x}_3 | \tilde{x}_4 | RHS |
|---------------|--|--|--|---|---|
| \tilde{z} | {(-11, -7, -3, 0; 0, 3, 7, 11), (-24, -11.5, -5.5, -2; 5, 7.5, 11, 18)} | {(-16, -9, -4, 0; 2, 6, 11, 18), (-24, -13, -6.5, -3.5; 5, 10.5, 14.5, 23.5)} | {(-5, -3, -1, 0; 0, 1, 3, 5), (-6, -3, -1.5, -1; 0, 1.5, 3.5, 6)} | {(-7, -5, -3, -2; -2, -1, 1, 3), (-8, -5, -3.5, -3; -2, -0.5, 1.5, 4)} | {(-13, -10, -7, -4; -4, -1, 2, 5), (-26, -18, -12.5, -6; -1, 3.5, 10.5, 18)} |
| RV | 0 | 1 | 0 | -2 | -4 |
| \tilde{x}_3 | {(-7, -5, -3, -1; -2, 2, 3.5, 5), (-13, -8, -5, -3; 2, 4, 8, 14)} | {(-11, -8, -4, -2; 0, 2, 6, 9), (-19, -12, -7, -4; 3, 7, 9, 14)} | 1 | 1 | {(-4, -2, 0, 4; 4, 8, 10, 12), (-20, -11, -6, 2; 8, 13, 20, 26)} |
| RV | 0 | -1 | | | 4 |
| \tilde{x}_1 | {(-2, -1, 0, 1; 1, 2, 3, 4), (-5, -3, -2, -1; 2, 3, 5, 10)} | {(-7, -5, -3, -2; -2, -1, 1, 3), (-12, -7, -5, -4; 0, 2, 3, 6)} | 0 | 1 | {(-2, -1, 0, 2; 2, 4, 5, 6), (-14, -9, -5, 1; 5, 8, 13, 18)} |
| RV | 1 | -2 | | | 2 |

6. CONCLUSION

We have considered the intuitionistic fuzzy linear programming (IFLP) with intuitionistic polygonal fuzzy numbers (IPFN). Most of the previous work in the literature are special cases of the IPFN such as triangular or rectangular IFN's and most of them convert the problem into a crisp and then solve with the traditional methods. The polygonal IFN's considered in this paper is a generalization of these existing IFN's. We have discussed how to modify the simplex method to solve IFLP with IPN problems without converting it to crisp. We showed how to use the simplex method in the tableau format and discuss the cases of the final solution based on the optimality conditions. We implement the modified simplex method to solve two examples one has exact solution and the other one has a unbounded solutions.

ACKNOWLEDGEMENT

This work was funded by the Deanship of Scientific Research in Jordan University of Science and Technology under research project number 20180486.





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