An enhanced fletcher-reeves-like conjugate gradient methods for image restoration

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ABSTRACT

Noise is an unavoidable aspect of modern camera technology, causing a decline in the overall visual quality of the images. Efforts are underway to diminish noise without compromising essential image features like edges, corners, and other intricate structures. Numerous techniques have already been suggested by many researchers for noise reduction, each with its unique set of benefits and drawbacks. Denoising images is a basic challenge in image processing. We describe a two-phase approach for removing impulse noise in this study. The adaptive median filter (AMF) for salt-and-pepper noise identifies noise candidates in the first phase. The second step minimizes an edge-preserving regularization function using a novel hybrid conjugate gradient approach. To generate the new improved search direction, the new algorithm takes advantage of two well-known successful conjugate gradient techniques. The descent property and global convergence are proven for the new methods. The obtained numerical results reveal that, when applied to image restoration, the new algorithms are superior to the classical fletcher reeves (FR) method in the same domain in terms of maintaining image quality and efficiency.

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1. INTRODUCTION

The subject of image restoration has been widely researched and applied in numerous fields of science and engineering. It requires the reconstruction of an original scene from a deteriorated observation. For example, air turbulence degrades star pictures viewed by ground-based telescopes. Images are often exposed to noise due to environmental factors, transmission channels, and other related elements during acquisition, compression, and transmission. As a result, the image quality is affected, leading to distortion and loss of image information. Noise also impacts later image processing tasks, such as the analysis and tracking of images, as well as video processing. Thus, image denoising is a crucial aspect of modern image processing systems.

Image denoising aims to restore the original image quality by minimizing noise from a noise-rich image. However, since noise, edge, and texture are high-frequency constituents, it is challenging to differentiate between them during denoising. As a result, restored images may lose some important details. Overall, the challenge in image processing systems is to recover relevant information from noisy images during noise removal to produce high-quality images.

There are certain cases where stellar pictures must be recovered even if they have not been observed within the atmosphere. The fundamental goal of this research is to create a class of iterative optimization algorithms applicable to edge-preserving regularization (EPR) objective functions. To reduce impulse noise, a two-phase technique was recently developed in [1]. For salt-and-pepper noise, the adaptive median filter (AMF) is used, while for random-valued noise, the adaptive center-weighted median filter (ACWMF) is used, which is first improved by applying the variable window technique to increase its detection capabilities in severely damaged pictures [2]. We only utilize the salt-and-pepper noise in this study. Let *X* represent the true picture and $\mathrm{A} = \{1,2,3,\ldots\ldots \mathrm{M}\} \times \{1,2,3,\ldots\ldots \mathrm{N}\}$ represent the index set of *X*. Let $\mathrm{N} \subset \mathrm{A}$ denote the set of noise pixel indices detected during the first phase. Also, let $\mathrm{P_{i,j}}$ be the set of pixel's four nearest neighbors at position $(\mathrm{i},\mathrm{j}) \in \mathrm{A}$, $\mathrm{y_{i,j}}$ denote the discovered pixel value of the actual picture at position (i,j), and $\mathrm{u_{i,j}} = \left[\mathrm{u_{i,j}}\right]_{(\mathrm{i},\mathrm{j})\in \mathrm{N}}$ denote a lexicographically ordered column vector of length *c*, where *c* represents the size of *N*. Then, minimizing the following function will recover the noise pixels.

$$f_\alpha(u) = \sum_{(i,j)\in N}\left[\left|u_{i,j} - y_{i,j}\right| + \frac{\beta}{2}(2 \times S_{i,j}^1 + S_{i,j}^2)\right] \tag{1}$$

where β is the regularization parameter,

$$S_{i,j}^1 = 2\sum_{(m,n)\in P_{i,j}\cap N^c} \phi_\alpha(u_{i,j} - y_{m,n}) \text{ and } S_{i,j}^2 = \sum_{(m,n)\in P_{i,j}\cap N} \phi_\alpha(u_{i,j} - y_{m,n}).$$

Function (1) is a hypothetical function that preserves the edges $\phi_\alpha = \sqrt{\alpha + x^2}, \alpha > 0$, that can be used to describe impulsive noise in general. Minimizing (1) defines the essence of a slavish AMF introduced in [3]. This is a typical method for locating pixels that may be contaminated. In practice, the non-smooth data-fitting term might be dropped because it is not needed in the second phase, when only poor-quality pixels are recovered after reduction. As a result, a number of optimization strategies for minimizing the following smooth EPR functional may be utilized (such as [4]–[7]).

$$f_\alpha(u) = \sum_{(i,j)\in N}\left[(2 \times S_{i,j}^1 + S_{i,j}^2)\right] \tag{2}$$

The conjugate gradient (CG) method for image correction is quite effective for solving unconstrained optimization problems of the form (3),

$$Min\, f(x)\,, \mathrm{x} \in R^n \tag{3}$$

due to their low memory requirements and simplicity of coding [4]–[7]. To solve (1), an iterative computation of a new solution vector is done using (4).

$$x_{k+1} = x_k + \alpha_k d_k \tag{4}$$

The step length α_k is calculated is traditionally obtained through a one-dimensional line search which, in practice, is usually inexact due to cost and impracticality considerations. For quadratic functions, α_k can be computed exactly using [8].

$$\alpha_k = \frac{-g_k^T d_k}{d_k^T Q d_k} \tag{5}$$

However, for general functions, α_k is computed to ensure that the obtained search direction is sufficiently downhill through satisfying the strong Wolfe conditions [2].

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^T d_k \tag{6a}$$

and

$$d_k^T g(x_k + \alpha_k d_k) \ge \sigma\, d_k^T g_k \tag{6b}$$

where $0 < \delta < \sigma < 1$. CG methods compute search directions using (7),

$$d_{k+1} = -g_{k+1} + \beta_k s_k \tag{7}$$

where β_k is a normally referred to as a conjugacy parameter and both d_k and d_{k+1} satisfy the conjugacy condition $d_i^T Q d_j = 0, \forall i \neq j$, for a symmetric matrix $Q \in R^{n\times n}$. A variety of equations have been published to compute the scalar β_k. Two well-known conjugate gradient approaches are Fletcher [9] and Dai and Yuan [10], with

$$\beta_k^{FR} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k} \text{ and } \beta_k^{DY} = \frac{g_{k+1}^T g_{k+1}}{d_k^T y_k} \tag{8}$$

respectively. The two methods have been the focus of many studies, not just because of their historical significance, but also because of their proven global convergence. Many other variants have been examined in an attempt to improve the numerical behavior of the CG methods, given their attractive storage requirements; see, for example, [11]–[14]. CG methods can be utilized in solving problems related to machine learning, fluid mechanics, solution of nonlinear equations and differential equations, deep learning, in addition to other applications. Another possible area of application is human performance technology (HPT). HPT is largely based on computer systems' numerical performance improvement (PI) characteristics, which rely on logical judgements enabled by specialized algorithms [15]. PI also helps to widen the scope of instructional design by using a systems perspective to address performance opportunities and obstacles. CG methods have proven valuable in solving issues in the adoption of mobile electronic performance support systems (EPSS) improved the job performance and efficiency of mobile users, according to a cross-sectional qualitative research [16].

In order to improve the computational efficiency of the standard CG method, a special type of conjugate gradient methods have recently been extensively investigated [17]–[20]. The approaches in [9], [18] propose a conjugacy condition of the form.

$$\beta_k^{HY} = \frac{\|g_{k+1}\|^2}{2/\alpha_k(f_k - f_{k+1})}, \beta_k^{B} = \frac{\|g_{k+1}\|^2}{(f_k - f_{k+1})/\alpha_k - g_k^T d_k/2} \tag{9}$$

Unlike the traditional CG algorithms, the aforementioned approach has the unique characteristic of consistently constructing better descent directions while satisfying the conjugacy conditions, as evidenced by the reported results. In the following section, a quadratic model will be exploited to derive new conjugacy parameters β_k, giving rise to new CG algorithms.

2. NEW CONJUGATE GRADIENT COEFFICIENTS

The formulation of the new CG method presented here exploits a classical quadratic model characterized by its simplicity. Iiduka and Narushima [21] propose the following choice for β_k.

$$\beta_k = \frac{g_{k+1}^T Q s_k}{d_k^T Q s_k} \tag{10}$$

where Q is the constant Hessian of some quadratic function. The parameter β_k satisfies a conjugacy condition of the form:

$$d_{k+1}^T Q d_k = 0 \tag{11}$$

In our derivation we will introduce an appropriate approximation to the quantity $d_k^T Q s_k$, essential to our proposed method. Assume f is a quadratic function of the form:

$$f_{k+1} = f_k + s_k^T g_k + \frac{1}{2} s_k^T Q(x_k) s_k. \tag{12}$$

This quadratic function's gradient is explicitly given by (13).

$$g_{k+1} = g_k + Q(x_{k+1}) s_k \tag{13}$$

As a result, curvature information may be expressed as (14).

$$s_k^T Q(x_k) s_k = 2 s_k^T y_k + 2(f_{k+1} - f_k) \tag{14}$$

From (14), we obtain:

$$d_k^T Q(x_k) s_k = 2\frac{\alpha_k (g_k^T d_k)^2}{(s_k^T y_k + 2(f_k - f_{k+1}))} \tag{15}$$

plugging (15) into (11), we get

$$\beta_k = \frac{g_{k+1}^T y_k}{2\alpha_k (g_k^T d_k)^2 / (s_k^T y_k + 2(f_k - f_{k+1}))} \tag{16}$$

given that conjugacy holds and exact line search, (16) becomes

$$\beta_k^{BL1} = \frac{g_{k+1}^T g_{k+1}}{2\alpha_k (g_k^T d_k)^2 / (s_k^T y_k + 2(f_k - f_{k+1}))} \tag{17}$$

or, alternatively,

$$\beta_k^{BL2} = \frac{g_{k+1}^T g_{k+1}}{2\alpha_k (g_k^T d_k)^2 / (-s_k^T g_k + 2(f_k - f_{k+1}))} \tag{18}$$

and

$$\beta_k^{BL3} = \frac{g_{k+1}^T g_{k+1}}{2\alpha_k (g_k^T d_k)^2 / (\alpha_k g_k^T g_k + 2(f_k - f_{k+1}))} \tag{19}$$

The new expressions for β_k are collectively referred to here as BL algorithms. The algorithmic framework is given next. BL algorithm:
Stage 1. Given $x_1 \in R^n$. Initialize $k = 1$ and $d_1 = -g_1$. If $\|g_1\| \le 10^{-6}$, then stop.
Stage 2. Compute $\alpha_k > 0$ satisfying conditions (6).
Stage 3. Compute $x_{k+1} = x_k + \alpha_k d_k$ and $g_{k+1} = g(x_{k+1})$. If $\|g_{k+1}\| \le 10^{-6}$, then terminate.
Stage 4. Evaluate β_k using $(17 - 19)$, then construct d_{k+1} by (7).
Stage 5. Set $k = k + 1$ and continue with step 2.
Theorem 1. The quantities $\{x_k\}$ and $\{d_k\}$, computed by the new methods satisfy.

$$d_{k+1}^T g_{k+1} < 0 \text{ and } d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k \tag{20}$$

Proof: If $d_k = -g_k$ then $d_1^T g_1 < 0$**.** Suppose that $d_k^T g_k < 0$ for any *k*. From (8) and (19), it is easy to show that

$$\begin{aligned} d_{k+1}^T g_{k+1} &= -g_{k+1}^T g_{k+1} + \beta_k d_k^T g_{k+1} \\ &= -\beta_k (2\alpha_k (g_k^T d_k)^2 / (s_k^T y_k + 2(f_k - f_{k+1}))) + \beta_k d_k^T g_{k+1} \end{aligned} \tag{21}$$

The following is the outcome of using the (22).

$$d_{k+1}^T g_{k+1} = \beta_k [d_k^T g_{k+1} - (2\alpha_k (g_k^T d_k)^2 / (s_k^T y_k + 2(f_k - f_{k+1})))] \tag{22}$$

We obtain, using (16) and (21),

$$d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k \tag{23}$$

From the downhill property of the search direction, it is obvious that $d_k^T g_k < 0$, thus we get

$$d_{k+1}^T g_{k+1} < 0 \tag{24}$$

The proof is thus complete.

3. CONVERGENCE ANALYSIS

In order to establish the global convergence of the BL algorithms, the following assumptions are needed:
- The level set $\Omega = \{x \in R^n / f(x) \le f(x_1)\}$ is bounded.
- In some neighborhood Λ of Ω, the gradient g of the objective function is Lipschitz continuous, namely, there exists some constant $L > 0$ such that

$$\|g(o) - g(\tau)\| \leq L\|o - \tau\|, \forall \tau, o \in \Lambda \tag{25}$$

(see [22] for more details). The theorems in [23] have proven to be useful in proving global convergence. We adopt some of those here and prove them for our methods and some of the results in [24], [25].
Lemma 1. Assume that assumptions 1 and 2 above hold. Then for any iteration a method that produces α_k by doing the Wolfe line search, the (26) holds.

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty \tag{26}$$

Theorem 2. Assume that Assumptions 1 and 2 above hold. If formula β_k satisfies (20), then we have:

$$\lim_{k\to\infty} inf \|g_k\| = 0 \tag{27}$$

Proof: By induction, assume that (27) does not hold. The (8) may be expressed as $d_{k+1} + g_{k+1} = \beta_k d_k$. Upon squaring both sides, we get

$$\|d_{k+1}\|^2 + \|g_{k+1}\|^2 + 2d_{k+1}^T g_{k+1} = (\beta_k)^2 \|d_k\|^2 \tag{28}$$

Using (23), the following results hold:

$$\|d_{k+1}\|^2 = \frac{(d_{k+1}^T g_{k+1})^2}{(d_k^T g_k)^2} \|d_k\|^2 - 2d_{k+1}^T g_{k+1} - \|g_{k+1}\|^2 \tag{29}$$

Upon dividing both sides of (29) by $(d_{k+1}^T g_{k+1})^2$, we get

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} = \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \frac{\|g_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} - \frac{2}{d_{k+1}^T g_{k+1}} \tag{30}$$

This yield

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} \leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \left(\frac{\|g_{k+1}\|}{(d_{k+1}^T g_{k+1})} + \frac{1}{\|g_{k+1}\|^2}\right) + \frac{1}{\|g_{k+1}\|^2} \leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} + \frac{1}{\|g_{k+1}\|^2} \tag{31}$$

Hence,

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} \leq \sum_{i=1}^{k+1} \frac{1}{\|g_i\|^2} \tag{32}$$

Assume that $c_1 > 0$ exists such that $\|g_k\| > c_1$ for every $k \in n$. Then

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} < \frac{k+1}{c_1^2} \tag{33}$$

We can see that, using the assumption and (33) as a guide,

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} = \infty \tag{34}$$

By Lemma 1, we may conclude that $\lim_{k\to\infty} \inf\|g_k\| = 0$ holds.

4. NUMERICAL RESULTS

The BL1, BL2, and BL3 algorithms performance is examined in the domain of minimizing salt-and-pepper impulse noise (3). The test images are listed in Table 1. Table 1 also reports the numerical results for comparing the classical fletcher reeves (FR) method to the newly derived ones in terms of the number of iterations, function/gradient evaluation count in addition to peak signal-to-noise ratio (PSNR). All simulations are run using MATLAB 2015a. It is worth emphasizing that the major focus of the study is on how fast the problem of reducing carbon emissions in (3) can be tackled efficiently. The pixel quality of the corrected pictures is assessed using PSNR value given by (35),

$$PSNR = 10\, log_{10} \frac{255^2}{\frac{1}{MN}\sum_{i,j}(u_{i,j}^r - u_{i,j}^*)^2} \tag{35}$$

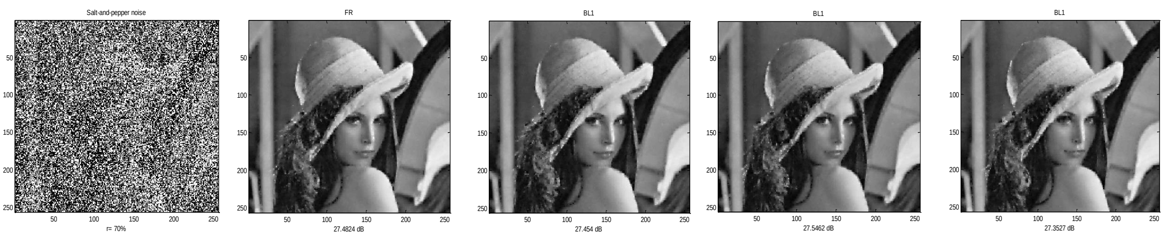
where $u_{i,j}^r$ and $u_{i,j}^*$ denote the pixel values of the corrected and the original image, respectively. For both procedures, the following are the termination conditions (36).

$$\frac{|f(u_k)-f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \text{ and } \|f(u_k)\| \leq 10^{-4}(1+|f(u_k)|) \tag{36}$$

Figures 1 to 4, show the obtained results by applying the algorithms to the noisy pictures. Figures 1(a), 2(a), 3(a) and 4(a) are the images corrupted with 70% salt-and-pepper noise; Figures 1(b), 2(b), 3(b) and 4(b) are results of the FR method; Figures 1(c), 2(c), 3(c) and 4(c) are results of the BL1 method; Figures 1(d), 2(d), 3(d) and 4(d) are results of the BL2 method; Figures 1(e), 2(e), 3(e) and 4(e) are results of the BL3 method. These results show that the suggested image correction methods BL1, BL2, and BL3 are both effective and efficient.

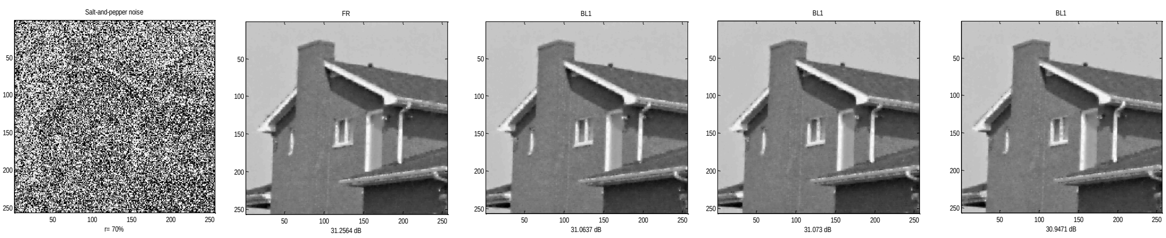
Table 1. Numerical results of FR and new algorithms

Image	Noise level r (%)	FR-Method			BL1-Method			BL2-Method			BL3-Method		
		NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)
Le	50	82	153	30.5529	41.0	89.0	30.7909	41.0	87.0	30.4785	44.0	95.0	30.5044
	70	81	155	27.4824	47.0	98.0	27.454	47.0	96.0	27.5462	44.0	90.0	27.3527
	90	108	211	22.8583	47.0	95.0	22.6477	52.0	102.0	23.0528	59.0	122.0	23.0546
Ho	50	52	53	30.6845	27.0	56.0	34.7784	28.0	57.0	35.0223	27.0	56.0	34.6089
	70	63	116	31.2564	35.0	71.0	31.1851	32.0	64.0	31.073	24.0	45.0	30.9471
	90	111	214	25.287	53.0	109.0	25.0672	53.0	105.0	25.0622	57.0	117.0	25.2418
El	50	35	36	33.9129	25.0	46.0	33.8945	26.0	48.0	33.8942	25.0	46.0	33.8902
	70	38	39	31.864	28.0	53.0	31.8199	28.0	51.0	31.835	30.0	56.0	31.8211
	90	65	114	28.2019	39.0	74.0	28.2269	38.0	70.0	28.266	41.0	80.0	28.154
c512	50	59	87	35.5359	28.0	59.0	35.4456	27.0	58.0	35.4694	30.0	62.0	35.5779
	70	78	142	30.6259	35.0	75.0	30.691	35.0	73.0	30.6525	27.0	55.0	30.6611
	90	121	236	24.3962	44.0	92.0	24.9656	48.0	103.0	24.8162	54.0	113.0	24.9088



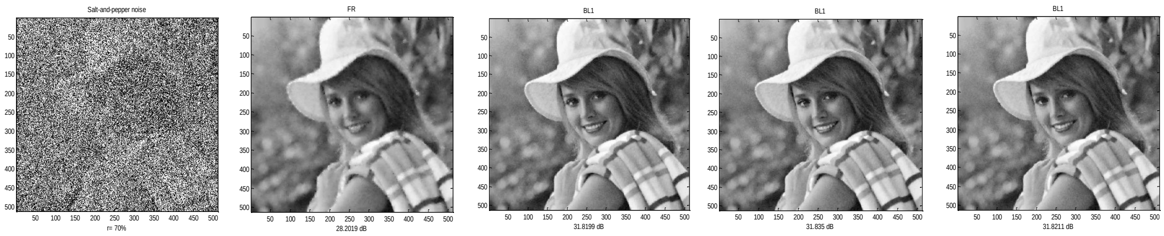
(a) (b) (c) (d) (e)

Figure 1. The noisy image and the corresponding corrected images results for each algorithm with 70% salt-and-pepper noise applied to 256*256 Lena image (a) original image, (b) FR output, (c) BL1 output, (d) BL12 output, and (e) BL3 output



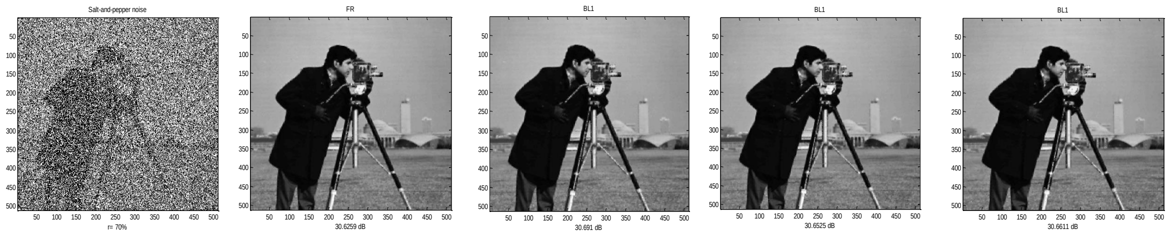
(a) (b) (c) (d) (e)

Figure 2. The noisy image and the corresponding corrected images results for each algorithm with 70% salt-and-pepper noise applied to 256×256 house image; (a) original image, (b) FR output, (c) BL1 output, (d) BL2 output, and (e) BL3 output



(a) (b) (c) (d) (e)

Figure 3. The noisy image and the corresponding corrected images results for each algorithm with 70% salt-and-pepper noise applied to 256×256 Elaine image (a) original image, (b) FR output, (c) BL1 output, (d) BL2 output, and (e) BL3 output



(a) (b) (c) (d) (e)

Figure 4. The noisy image and the corresponding corrected images results for each algorithm with 70% salt-and-pepper noise applied to 256×256 cameraman image (a) original image, (b) FR output, (c) BL1 output, (d) BL2 output, and (e) BL3 output

5. CONCLUSION

The focus of this research has been on the creation of novel, modified conjugate gradient formulae that outperform the traditional FR CG approach for picture restoration. The results confirm to the effectiveness of the strategy used in this research to derive variations of the traditional CG technique. The novel techniques have demonstrated global convergence under the rigorous Wolfe line search conditions. The testing simulations have demonstrated that, in the majority of instances, the novel approaches, BL1, BL2, and BL3, significantly reduce iteration counts and function evaluations while maintaining the same picture restoration quality.

REFERENCES

[1] Y. Dai, J. Han, G. Liu, D. Sun, H. Yin, and Y.-X. Yuan, "Convergence properties of nonlinear conjugate gradient methods," *SIAM Journal on Optimization*, vol. 10, no. 2, pp. 345–358, Jan. 2000, doi: 10.1137/S1052623494268443.

[2] A. Antoniou and W.-S. Lu, *Practical optimization*. Boston, MA: Springer US, 2007, doi: 10.1007/978-0-387-71107-2.

[3] W. Xue, J. Ren, X. Zheng, Z. Liu, and Y. Liang, "A new DY conjugate gradient method and applications to image denoising," *IEICE Transactions on Information and Systems*, vol. E101.D, no. 12, pp. 2984–2990, 2018, doi: 10.1587/transinf.2018EDP7210.

[4] S. M. Ibrahim, U. A. Yakubu, and M. Mamat, "Application of spectral conjugate gradient methods for solving unconstrained optimization problems," *An International Journal of Optimization and Control: Theories and Applications (IJOCTA)*, vol. 10, no. 2, pp. 198–205, Jun. 2020, doi: 10.11121/ijocta.01.2020.00859.

[5] X. Zheng and J. Shi, "A modified sufficient descent Polak-Ribiére-Polyak type conjugate gradient method for unconstrained optimization problems," *Algorithms*, vol. 11, no. 9, p. 133, Sep. 2018, doi: 10.3390/a11090133.

[6] B. A. Hassan, H. O. Dahawi, and A. S. Younus, "A new kind of parameter conjugate gradient for unconstrained optimization," *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS)*, vol. 17, no. 1, pp. 404–411, Jan. 2020, doi: 10.11591/ijeecs.v17.i1.pp404-411.

[7] B. A. Hassan, "A new type of quasi-newton updating formulas based on the new quasi-newton equation," *Numerical Algebra, Control & Optimization*, vol. 10, no. 2, pp. 227–235, 2020, doi: 10.3934/naco.2019049.

[8] J. Nocedal and S. J. Wright, *Numerical optimization*. Springer New York, 2006, doi: 10.1007/978-0-387-40065-5.

[9] R. Fletcher, "Function minimization by conjugate gradients," *The Computer Journal*, vol. 7, no. 2, pp. 149–154, Feb. 1964, doi: 10.1093/comjnl/7.2.149.

[10] Y. H. Dai and Y. Yuan, "A nonlinear conjugate gradient method with a strong global convergence property," *SIAM Journal on Optimization*, vol. 10, no. 1, pp. 177–182, Jan. 1999, doi: 10.1137/S1052623497318992.

[11] M. R. Hestenes and E. Stiefel, "Methods of conjugate gradients for solving linear systems," *Journal of Research of the National Bureau of Standards*, vol. 49, no. 6, 1952, doi: 10.6028/jres.049.044.

[12] M. J. D. Powell, "Nonconvex minimization calculations and the conjugate gradient method," in *Lecture Notes in Mathematics*, Springer Berlin Heidelberg, 1984, pp. 122–141, doi: 10.1007/BFb0099521.

[13] Y. Liu and C. Storey, "Efficient generalized conjugate gradient algorithms, part 1: Theory," *Journal of Optimization Theory and Applications*, vol. 69, no. 1, pp. 129–137, Apr. 1991, doi: 10.1007/BF00940464.

[14] E. Polak and G. Ribiere, "Note on the convergence of conjugate direction methods (in French Note sur la convergence de méthodes de directions conjuguées)," *Revue française d'informatique et de recherche opérationnelle. Série rouge*, vol. 3, no. 16, pp. 35–43, 1969, doi: 10.1051/m2an/196903R100351.

[15] I. M. Sulaiman, M. Mamat, A. E. Owoyemi, P. L. Ghazali, M. Rivaie, and M. Malik, "The convergence properties of some descent conjugate gradient algorithms for optimization models," *Journal of Mathematics and Computer Science*, vol. 22, no. 3, pp. 204–215, Jul. 2020, doi: 10.22436/jmcs.022.03.02.

[16] A. Awasthi and H. Omrani, "A goal-oriented approach based on fuzzy axiomatic design for sustainable mobility project selection," *International Journal of Systems Science: Operations and Logistics*, vol. 6, no. 1, pp. 86–98, Jan. 2019, doi: 10.1080/23302674.2018.1435834.

[17] A. B. Abubakar, P. Kumam, M. Malik, P. Chaipunya, and A. H. Ibrahim, "A hybrid FR-DY conjugate gradient algorithm for unconstrained optimization with application in portfolio selection," *AIMS Mathematics*, vol. 6, no. 6, pp. 6506–6527, 2021, doi: 10.3934/math.2021383.

[18] D. Touati-Ahmed and C. Storey, "Efficient hybrid conjugate gradient techniques," *Journal of Optimization Theory and Applications*, vol. 64, no. 2, pp. 379–397, Feb. 1990, doi: 10.1007/BF00939455.

[19] I. A. Moghrabi, "Metric-based parameterizations for multi-step unconstrained optimization," *Computational Optimization and Applications*, vol. 19, no. 3, pp. 337–345, 2001.

[20] F. N. Jardow and G. M. Al-Naemi, "A new hybrid conjugate gradient algorithm for unconstrained optimization with inexact line search," *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS)*, vol. 20, no. 2, pp. 939–947, Nov. 2020, doi: 10.11591/ijeecs.v20.i2.pp939-947.

[21] H. Iiduka and Y. Narushima, "Conjugate gradient methods using value of objective function for unconstrained optimization," *Optimization Letters*, vol. 6, no. 5, pp. 941–955, Jun. 2012, doi: 10.1007/s11590-011-0324-0.

[22] J. Mo, N. Gu, and Z. Wei, "Hybrid conjugate gradient methods for unconstrained optimization," *Optimization Methods and Software*, vol. 22, no. 2, pp. 297–307, Apr. 2007, doi: 10.1080/10556780500518653.

[23] G. Zoutendijk, "Some algorithms based on the principle of feasible directions," in *Nonlinear Programming*, Elsevier, 1970, pp. 93–121, doi: 10.1016/B978-0-12-597050-1.50008-7.

[24] J. Zhang and C. Xu, "Properties and numerical performance of quasi-Newton methods with modified quasi-Newton equations," *Journal of Computational and Applied Mathematics*, vol. 137, no. 2, pp. 269–278, 2001, doi: 10.1016/S0377-0427(00)00713-5.

[25] N. Andrei, "Open problems in nonlinear conjugate gradient algorithms for unconstrained optimization," *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, vol. 34, no. 2, pp. 319–330, 2011.

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