Implementation of variational iteration method for various types of linear and nonlinear partial differential equations

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Article Info

Article history:

Received Mar 16, 2022
Revised Sep 24, 2022
Accepted Oct 20, 2022

Keywords:

Airy equation
Benjamin equation
Lagrange multiplier
Partial differential equations
Potential Korteweg-De-Vries
Variational iteration method

ABSTRACT

There are various linear and nonlinear one-dimensional partial differential equations that are the focus of this research. There are a large number of these equations that cannot be solved analytically or precisely. The evaluation of nonlinear partial differential equations, even if analytical solutions exist, may be problematic. Therefore, it may be necessary to use approximate analytical methodologies to solve these issues. As a result, a more effective and accurate approach must be investigated and analyzed. It is shown in this study that the Lagrange multiplier may be used to get an ideal value for parameters in a functional form and then used to construct an iterative series solution. Linear and nonlinear partial differential equations may both be solved using the variational iteration method (VIM) method, thanks to its high computing power and high efficiency. Decoding and analyzing possible Korteweg-De-Vries, Benjamin, and Airy equations demonstrates the method's ability. With just a few iterations, the produced findings are very effective, precise, and convergent to the exact answer. As a result, solving nonlinear equations using VIM is regarded as a viable option.

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1. INTRODUCTION

There are numerous fields of science and engineering where nonlinear phenomena are fundamental. There are numerous fields of science and engineering where nonlinear phenomena are of fundamental importance, and this is no exception. Neither numerically nor analytically, it is still difficult to solve the nonlinear models of real-world problems [1]. The study of partial differential equations began in the eighteenth-century AD with a group of researchers such as Dalembert, Euler, and Lagrange such as issues related to heat, sound, elasticity, and fluid flow. Linear and nonlinear partial differential equations (PDE) are significant in many domains, including science and engineering, chemical reaction, fluid dynamics, nonlinear optics, dispersion, and plasma physics. Not all PDE problems in real-world models can be simply solved using differential equations. As a result, rather than solving those PDEs analytically, the optimal result can be obtained numerically or approximately. Also, we seek to obtain more accurate solutions for these problems which have great effectiveness in real life systems, such as the Adomian decomposition method (ADM) [2], [3], differential transform method (DTM) [4]–[6], homotropy perturbation method (HPM) [7]–[9], the local meshless method (LMM) [10]–[15], variational iteration method (VIM) [16], [17], the fractional iterative algorithm [18], modified variational iteration algorithm-I (mVIA-I) [19], modified variational iteration algorithm [20], [21], and the modified Laplace variational iteration method [22].

The VIM was initially presented [23]–[26] in 1998. It has been used by many technologists to solve mathematical and physical problems due to its ability to reduce a complex problem to an easy one. It was chosen better than numerical methods because it is free from errors and can solve a wide range of equations (ordinary equations, partial equations, and integral and differential equations), whether they are linear or non-linear [27]–[29].

Through research studies on this method, researchers have proven its importance and accuracy with the results, as well as its ability to reach an actual solution for most of the studied equations [30]–[34]. The variational iteration approach eliminates the drawbacks of the Adomian method. This method is a variant of the iteration method's general Lagrange multiplier method. Because it is easy to find solutions and is very accurate, its applications have grown.

Narayanamoorthy and Mathankumar [35] have claimed that VIM is more robust than other analytical methodologies like DTM and HPM. In contrast to HPM and DTM, which frequently use computer methods for nonlinear terms, VIM is used explicitly with no nonlinear term needs or restricted assumptions. Thus, with VIM, the series solutions obtained have a higher polynomial degree than those obtained by decomposition and perturbation methods.

The variational iteration method (VIM) solves differential equations that can change the structure of solutions without making any restrictive assumptions. The calculator in VIM is essential and accessible [36]. The VIM eliminates the difficulties of measuring Adomian polynomials in ADM [37], giving it a considerable advantage over ADM. The VIM also eliminates the need for discretization in numerical methods, resulting in an approximate solution with high accuracy, minimal calculation, and no physically unreasonable assumptions. This research aims to examine and analyze the approximate solution by the approximate analytical method based on VIM to the type of Korteweg-De-Vries, Benjamin, and Airy equations. Also, it would be of interest to investigate the feasibility and accuracy of the proposed method compared with other existing methods.

This paper is organized: in section 2, we present the mathematical formula for the VIM. In sections 3 to 5, we explained the applications of the method and how to create convergent solutions from exact solutions to the potential Korteweg-De-Vries, Benjamin, and Airy equations. Finally, the conclusions are in section 6.

2. MATHEMATICAL FORMULATION FOR VIM

To clarify the basic concepts of the VIM, we consider (1),

$$Lu(x,t) + Nu(x,t) = g(x,t) \tag{1}$$

where (*L*) is a linear operator, (*N*) is a nonlinear operator, and *g(t)* is an inhomogeneous term. A functional correction, according to the VIM, can be made:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[L\left(u_n(x,\xi)\right) + N\left(\widetilde{u_n}(x,\xi)\right) - g(x,\xi)\right] d\xi \qquad n \geq 0 \tag{2}$$

where λ is a Lagrange factorial that can be optimally calculated using the covariance theory. The abbreviation u_n refers is an approximate solution of *n*, and $\widetilde{u_n}$ is a limited variant, i.e., $\delta\widetilde{u_n} = 0$. So, we must first identify the Lagrange multiplier that will be ideally determined by integration by parts. Using the resulting Lagrange multiplier and any selected function u_0, the successive approximations $u_{n+1}(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be easily obtained. Therefore, the solution is (3).

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \tag{3}$$

3. ANALYSIS OF VIM FOR EQUATION FOR POTENTIAL KORTEWEG-DE-VRIES

We look at how successfully potential Korteweg-De-Vries equation (p-KDV) approximation solutions are implemented and checked using the method that we suggested in section 2. In order to accomplish this, we make use of the nonlinear p-KDV as a physical problem [37] in (4).

$$u_t + a(u_x)^2 + bu_{xxx} = 0 \tag{4}$$

where $u\ (x,t)$ represents the dependent variable, x and t represent the independent variables, and the parameters a and b are nonzero real constants. The equation's dark (topological) soliton solutions are provided by (5).

$$u(x,t) = A\ tanh[B(x - vt)] \tag{5}$$

where v denotes velocity and

$$A = \frac{6bB}{a} \quad , \quad B = \frac{1}{2}\sqrt{\frac{v}{b}} \tag{6}$$

Consider potential Korteweg-De-Vries (4) subject to the initial condition:

$$u(x,0) = A\ tanh(Bx) \tag{7}$$

The correction functional for (4) is in (8).

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial}{\partial\xi}u_n(x,\xi) + a\left(\frac{\partial}{\partial x}\widetilde{u_n}(x,\xi)\right)^2 + b\left(\frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right)\right]d\xi \tag{8}$$

where $\tilde{u}_n$ is restricted variation $\delta\widetilde{u_n} = 0$, $u_0(x,t)$ is an initial approximation or trial function and $\lambda(\xi)$ is a Lagrange multiplier. With the above correction functional stationary, we have:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\xi)\left[\frac{\partial}{\partial\xi}u_n(x,\xi) + a\left(\frac{\partial}{\partial x}\widetilde{u_n}(x,\xi)\right)^2 + b\left(\frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right)\right]d\xi \tag{9}$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\xi)\left[\frac{\partial}{\partial\xi}u_n(x,\xi)\right]d\xi \tag{10}$$

From this we get the stability condition:

$$\delta u_n : 1 + \lambda(\xi) = 0 \tag{11}$$

$$\delta u_n : \lambda'(\xi) = 0 \tag{12}$$

Therefore, the Lagrange multiplier can be identified as (13)

$$\lambda(\xi) = -1 \tag{13}$$

As a result, we can obtain the iteration (14).

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial}{\partial\xi}u_n(x,\xi) + a\left(\frac{\partial}{\partial x}u_n(x,\xi)\right)^2 + b\left(\frac{\partial^3}{\partial x^3}u_n(x,\xi)\right)\right]d\xi \tag{14}$$

Then, using the variational iteration (14), we begin with the initial approximation.

$$u_0(x,t) = u(x,0) = A\ tanh(Bx)$$

$$\begin{aligned} u_1(x,t) = A\ tanh(B\ x) - \ a\ A^2B^2t\ +\ 2\ aA^2B^2\ tanh^2(B\ x)\ t\ -\ aA^2B^2\ tanh^4(B\ x)\ t \\ -\ 8\ b\ AB^3\ tanh^2(B\ x)\ t\ +\ 6\ b\ AB^3\ tanh^4(B\ x)\ t\ +\ 2\ b\ A\ B^3 t \end{aligned}$$

Similarly, the remaining components of the iteration (14) can be derived using Maple software. We set $a = 1$, $b = 1$, and $v = 0.5$ to test the correctness and dependability of the VIM solution for the p-KdV equation. The 4-order approximation solution derived by VIM compared to the exact solution and the reduced differential transform method (RDTM) solution [37] is summarized in Table 1 and Figures 1 and 2 for different values of x, $t \in [0,1]$. The results we obtained are very close to the exact solution as well as the RDTM solution. According to Tables 1 and Figures 1 and 2, the 4-order VIM approximate solution satisfies the initial condition of p-KdV equation $a = b = 1\ and\ v = 0.5$ with sufficient accuracy compared with the exact solution RDTM [37] of p-KdV equation.

Table 1. Comparison between the solutions of the approximate obtained by VIM and exact solutions, also a comparison between absolute errors in VIM and RDTM of (4), at $a = b = 1$ and $v = 0.5$

x\t	Exact solutions	VIM solution $u_4(x,t)$	Absolute error (VIM) $\lvert u(x,t) - u_4(x,t)\rvert$	Absolute error (RDTM) $\lvert u(x,t) - u_4(x,t)\rvert$
0.10	0.03749609425	0.03749606573	2.852×10^{-10}	4.839076×10^{-10}
0.25	0.09368901252	0.09368693050	$2.082035141 \times 10^{-8}$	4.505072×10^{-8}
0.50	0.1870132399	0.1870085540	4.685844×10^{-6}	1.204876×10^{-6}
0.75	0.2796135561	0.2800088572	3.953011×10^{-6}	6.523635×10^{-6}
1.00	0.3711419684	0.3734682012	2.3262328×10^{-6}	1.481861×10^{-5}
2.00	0.7202372565	0.6889158600	$3.13213965 \times 10^{-4}$	6.634832×10^{-4}
3.00	1.030183947	1.054746570	2.456277×10^{-4}	4.331578×10^{-3}
4.00	1.291585757	1.4121069	1.205169×10^{-3}	6.034309×10^{-3}
5.00	1.502657418	1.37114764	1.316×10^{-2}	3.194238×10^{-4}

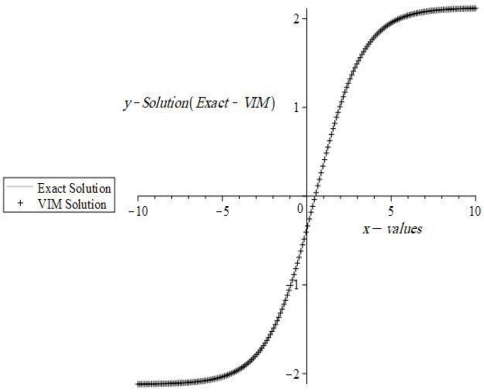

Figure 1. Exact and 4-order VIM approximate solution of p-KdV equation at $a = b = 1$ and $v = 0.5$ at $t = 1$

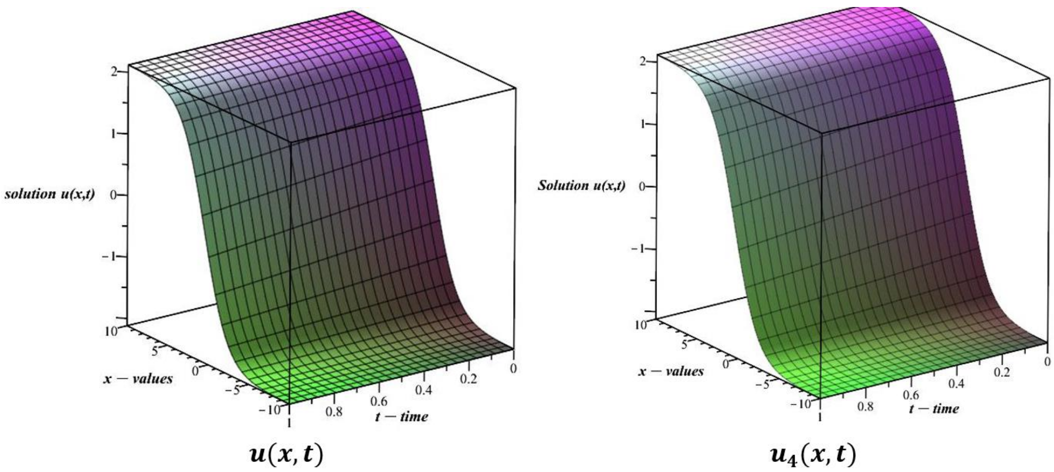

Figure 2. The exact and 4-order VIM approximate solution of p-KdV equation for $a = b = 1$ and $v = 0.5$

4. ANALYSIS OF VIM FOR BENJAMIN EQUATION

We investigate the extent to which the method that was proposed in section 2 for implementing and verifying approximation solutions for the Benjamin equation is utilized successfully. In order to achieve this goal, we make use of the nonlinear Benjamin equation that has been provided for you below as a physical issue [37].

$$u_{tt} + \alpha(uu_x)_x + \beta u_{xxxx} = 0 \tag{15}$$

where $u(x,t)$ is the dependent variable, and x and t are independent variables. The parameters α and β are real constants. The bright (non-topological) soliton solutions to the equation are provided by (16),

$$u(x,t) = A\, sech^2[B(x - vt)] \tag{16}$$

where v is velocity and

$$A = \frac{12\beta B^2}{\alpha}, \quad B = \frac{1}{2}\frac{v}{\sqrt{-\beta}} \tag{17}$$

We consider the Benjamin equation subject to initial condition:

$$u(x,0) = A\, sech^2(Bx) \tag{18}$$

$$u_t(x,0) = 2ABv\, sech^2(Bx) tanh(Bx) \tag{19}$$

The correction functional for (4) is in (20),

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left[\frac{\partial^2}{\partial \xi^2} u_n(x,\xi) + \alpha(\frac{\partial}{\partial x}(\widetilde{u_n}(x,\xi)\ \frac{\partial}{\partial x}\widetilde{u_n}(x,\xi))) + \beta\left(\frac{\partial^4}{\partial x^4}\widetilde{u_n}(x,\xi)\right)\right] d\xi \tag{20}$$

where $\widetilde{u_n}$ is restricted variation $\delta\widetilde{u_n} = 0$, $u_0(x,t)$ is an initial approximation or trial function and $\lambda(\xi)$ is a Lagrange multiplier. With the above correction functional stationary, we have:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda \left[\frac{\partial^2}{\partial \xi^2} u_n(x,\xi) + \alpha(\frac{\partial}{\partial x}(\widetilde{u_n}(x,\xi)\ \frac{\partial}{\partial x}\widetilde{u_n}(x,\xi))) + \beta\left(\frac{\partial^4}{\partial x^4}\widetilde{u_n}(x,\xi)\right)\right] d\xi \tag{21}$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\xi)\left[\frac{\partial^2}{\partial \xi^2} u_n(x,\xi)\right] d\xi \tag{22}$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta(\lambda \frac{\partial}{\partial \xi} u_n(x,\xi)) - \delta(\lambda' u_n(x,\xi)) - \delta\int_0^t \lambda'' u_n(x,\xi) d\xi \tag{22}$$

By using the stationary conditions:

$$\delta u_n : \lambda'' - \lambda = 0 \tag{24}$$

$$\delta u_n : 1 - \lambda' = 0 \tag{25}$$

$$\delta u'_n : \lambda = 0 \tag{26}$$

$$\delta u_n : \lambda'' = 0 \tag{27}$$

Therefore, the Lagrange multiplier can be identified as (28).

$$\lambda(\xi) = \xi - t \tag{28}$$

As a result, we can obtain the iteration in (29).

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \xi - t\left[\frac{\partial^2}{\partial \xi^2} u_n(x,\xi) + \alpha(\frac{\partial}{\partial x}(\widetilde{u_n}(x,\xi)\ \frac{\partial}{\partial x}\widetilde{u_n}(x,\xi))) + \beta\left(\frac{\partial^4}{\partial x^4}\widetilde{u_n}(x,\xi)\right)\right] d\xi \tag{29}$$

Then, using the variational iteration (29), we begin with the initial approximation.

$u(x,0) = A\, tanh(Bx)$, it follows that

$$u_0(x,t) = u(x,0) + t u_t(x,0)$$
$$u_0(x,t) = A\,\mathrm{sech}^2(Bx) + 2tABv\,\mathrm{sech}^2(Bx)\tanh(Bx)$$
$$\begin{aligned}
u_1(x,t) = {} & + A\,\mathrm{sech}^2(B\,x) + 2\,t\,A\,B\,v\,\mathrm{sech}^2(B\,x)\tanh(B\,x) - \frac{1}{3}\,\alpha t^4 A^2 B^4 v^2\,\mathrm{sech}^4(B\,x) \\
& - 5\,t^2\,\alpha A^2\,\mathrm{sech}^4(B\,x)\,B^2\tanh^2(B\,x) - 60\,t^2\beta\,A\,\mathrm{sech}^2(B\,x)\,B^4\tanh^4(B\,x) \\
& + 60\,t^2\beta\,A\,\mathrm{sech}^2(B\,x)\,B^4\tanh^2(B\,x) + t^2\,\alpha A^2\,\mathrm{sech}^4(B\,x)\,B^2 \\
& - 8\,t^2\beta\,A\,\mathrm{sech}^2(B\,x)\,B^4 - 7\,\alpha t^4 A^2 B^4 v^2\,\mathrm{sech}^4(B\,x)\tanh^4(B\,x) \\
& + \frac{14}{3}\,\alpha t^4 A^2 B^4 v^2\,\mathrm{sech}^4(B\,x)\tanh^2(B\,x) - 10\,t^3\,\alpha A^2\,\mathrm{sech}^4(B\,x)\,B^3\tanh^3(B\,x)\,v \\
& + \frac{14}{3} t^3\,\alpha A^2\,\mathrm{sech}^4(B\,x)\,B^3\tanh(B\,x)\,v - 120\,t^3\beta\,A\,B^5 v\,\mathrm{sech}^2(B\,x)\tanh^5(B\,x) \\
& + 160\,t^3\beta\,A\,B^5 v\,\mathrm{sech}^2(B\,x)\tanh^3(B\,x) - \frac{136}{3} t^3\beta\,A\,B^5 v\,\mathrm{sech}^2(B\,x)\tanh(B\,x)
\end{aligned}$$

In a similar fashion, the rest of the components of the iteration (29) can be obtained by using Maple software.

We set ($\alpha = -1,\ \beta = -3\ and\ v = 0.25$) to test the correctness and dependability of the VIM solution for the Benjamin equation. The 4-order approximation solution derived by VIM compared to the exact solution and the RDTM solution [37] is summarized in Table 1 and Figures 1 and 2 for different values of $x, t \in [0,1]$. The results we got were close to the exact solution, in addition to being better and more accurate than RDTM. According to Table 2 and Figures 3 and 4, the 4-order VIM approximate solution satisfies the initial condition of Benjamin equation $\alpha = -1,\ \beta = -3$ and $v = 0.25$ with better accuracy compared with RDTM [37] and the exact solution of Benjamin equation.

Table 2. The numerical results for the 4-approximation solution obtained by VIM in comparison with the exact solutions of (15) and the RDTM solution at $\alpha = -1,\ \beta = -3$ and $v = 0.25$

x\t	Exact solutions	VIM Solution $u_4(x,t)$	RDTM Solution $u_4(x,t)$	Absolute Error (VIM) $\lvert u(x,t) - u_4(x,t)\rvert$	Absolute Error (RDTM) $\lvert u(x,t) - u_4(x,t)\rvert$
0.10	0.1874945069	0.1874945069	0.1874961695	1.10×10^{-10}	1.662562×10^{-6}
0.25	0.1874656719	0.1874656719	0.1874744689	$5.645801890 \times 10^{-11}$	8.796962×10^{-6}
0.50	0.1873627379	0.1873627380	0.1873871610	$8.318579018 \times 10^{-11}$	2.442303×10^{-5}
0.75	0.1871913487	0.1871913486	0.1872217567	2.72945×10^{-11}	3.040799×10^{-5}
1.00	0.1869517547	0.1869517548	0.1869617528	4.62×10^{-11}	9.998060×10^{-6}
2.00	0.1853197872	0.1853197871	0.1846516678	$4.036710999 \times 10^{-11}$	6.681194×10^{-5}
3.00	0.1826417751	0.1826417747	0.1796230788	$3.194813670 \times 10^{-10}$	3.018696×10^{-3}
4.00	0.1789784742	0.1789784710	0.1712874217	3.2×10^{-9}	7.691052×10^{-3}
5.00	0.1744108363	0.1744108342	0.1596662760	$2.000290866 \times 10^{-9}$	1.474456×10^{-2}

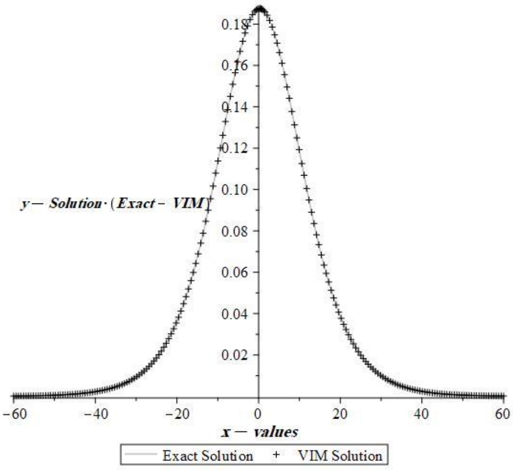

Figure 3. The graph 2D exact and 4-approximation solution of Benjamin equation for $\alpha = -1$, $\beta = -3$ and $v = 0.25\ at\ t = 1$

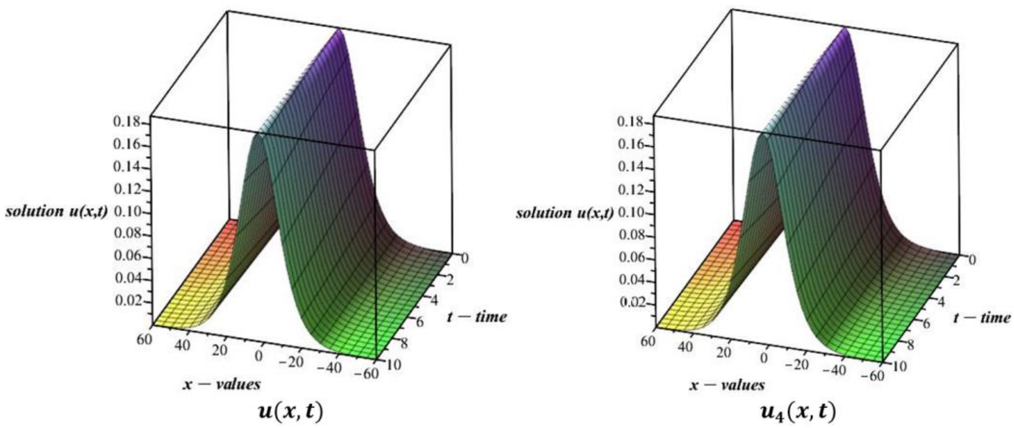

Figure 4. The graph 3D exact and 4-approximation solution of Benjamin equation for $\alpha = -1$, $\beta = -3$ and $v = 0.25$

5. APPLYING VIM FOR AIRY EQUATION

We conducted an investigation into the extent to which the method that was proposed in section 2 for putting into practice and verifying approximation solutions for the Airy equation is successfully utilized. We make use of the linear Airy equation, which has been laid out for your perusal down below as a physical issue [38], in order to accomplish this objective.

$$u_t + u_{xxx} = 0 \tag{30}$$

with the initial condition as in (31),

$$u(x,0) = sin(x) \tag{31}$$

the exact solution is given by (32).

$$u(x,0) = sin(x+t) \tag{32}$$

The correction functional for (30) is in (33):

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial}{\partial\xi}u_n(x,\xi) + \frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right]d\xi \tag{33}$$

where $\tilde{u}_n$ is restricted variation $\delta\widetilde{u_n} = 0$, $u_0(x,t)$ is an initial approximation or trial function and $\lambda(\xi)$ is a Lagrange multiplier. With the above correction functional stationary, we have:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\xi)\left[\frac{\partial}{\partial\xi}u_n(x,\xi) + \frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right]d\xi \tag{34}$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\xi)\left[\frac{\partial}{\partial\xi}u_n(x,\xi)\right]d\xi \tag{35}$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t)(1+\lambda(\xi)) - \delta\int_0^t \lambda'(\xi)u_n(x,\xi)d\xi \tag{36}$$

From this we get the stability condition:

$$\delta u_n : 1 + \lambda(\xi) = 0 \tag{37}$$

$$\delta u_n : \lambda'(\xi) = 0 \tag{38}$$

Therefore, the Lagrange multiplier can be identified as (39).

$$\lambda(\xi) = -1 \tag{39}$$

As a result, we can obtain the iteration in (40).

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial}{\partial\xi}u_n(x,\xi) + \frac{\partial^3}{\partial x^3}\widetilde{u_n}(x,\xi)\right]d\xi \tag{40}$$

Then, using the variational iteration (40), we begin with the initial approximation.

$$\begin{aligned}
&u_0(x,t) = u(x,0) = sin(x)\\
&u_1(x,t) = sin(x) + cos(x)\,t\\
&u_2(x,t) = sin(x) + cos(x)\,t - sin(x)\,t^2/2\\
&u_3(x,t) = sin(x) + cos(x)\,t - sin(x)\,t^2/2 - cos(x)\,t^3/6
\end{aligned}$$

Likewise, the remaining components of the iteration (40) can be derived by using maple. The 20-approximation solution derived by VIM and the exact solution are summarized in Table 3 and Figures 5 and 6 for different values of $x, t \in [0,1]$. According to Table 3 and Figures 5 and 6, the 20-order VIM approximate solution satisfies the initial condition of Airy equation with sufficient accuracy compared with the exact solution of Airy equation.

Table 3. The numerical results for the approximate solutions obtained by VIM in comparison with the exact solutions of (30)

x	Time (t)	Exact solutions	VIM- u_{20}	Absolute Error
0.1	0.1	0.198669330795061	0.198669330795062	9×10^{-16}
0.2	0.2	0.389418342308650	0.389418342308650	0
0.3	0.3	0.564642473395035	0.564642473395037	2×10^{-15}
0.4	0.4	0.717356090899523	0.717356090899524	1×10^{-15}
0.5	0.5	0.841470984807897	0.841470984807902	5×10^{-15}
0.6	0.6	0.932039085967226	0.932039085967228	2×10^{-15}
0.7	0.7	0.985449729988460	0.985449729988460	0
0.8	0.8	0.999573603041505	0.999573603041509	4×10^{-15}
0.9	0.9	0.973847630878195	0.973847630878200	5×10^{-15}
1.0	1.0	0.909297426825682	0.909297426825680	2×10^{-15}

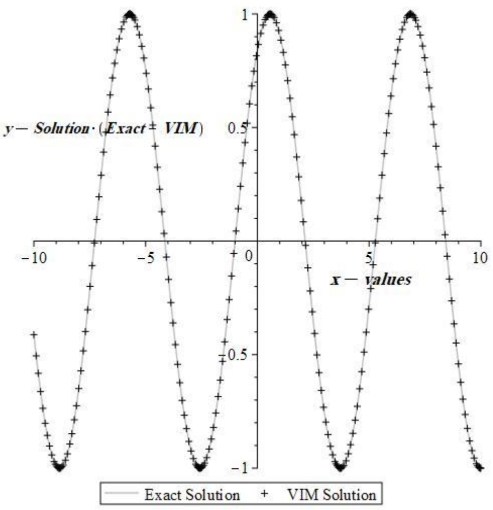

Figure 5. The graph 2D exact and approximate solution of Airy equation for $-10 \leq x \leq 10$

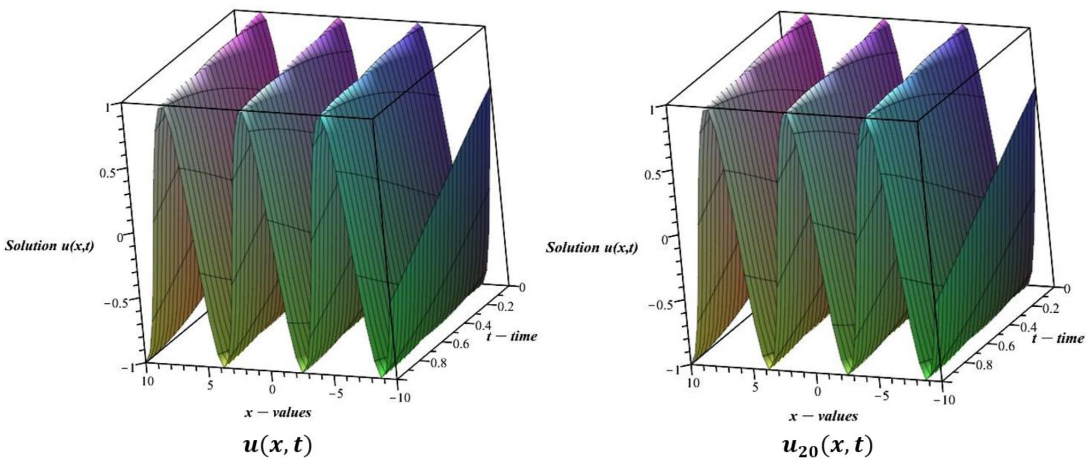

Figure 6. The graph 3D exact and approximate solution of Airy equation conformable to the values $-10 \leq x \leq 10,\ 0 \leq t \leq 1$

6. CONCLUSION

In this research, an approximate analytical method was introduced to solve partial differential equations in parabolic form. A scheme based on VIM to approximate the solution of the Korteweg-De-Vries equation, the Benjamin equation, and the Airy equation is analyzed and implemented. The research demonstrates that VIM can be implemented quickly without deconstructing the nonlinear variables in the given test problem. The accuracy of VIM is determined through Lagrange multiplier analysis, which generates a series of solutions that are convergent to the exact solution, assisting an engineer or scientist in gaining a better understanding of a physical problem and may contribute to the improvement of future techniques and designs utilized to tackle their challenges. The nonlinear test problems showed good accuracy when compared with the exact solution. Due to its accuracy and compliance with the VIM, it appears to be a reliable method for solving parabolic partial differential equations. In the future, we will use this method to solve problems with elliptic and hyperbolic models.

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