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Algorithm for solving fractional partial differential equations using homotopy analysis method with Padé approximation

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ABSTRACT

In recent years nonlinear problems have several methods to be solved and utilize a well-known analytic tools such as homotopy analysis method. In general, homotopy analysis method had gain a wide focus and improvement especially in typical nonlinear problem. The aim of this paper is to use homotopy method of analysis to solve partial differential equation in addition to improve method's efficiency. The method in this paper is to apply approximation to Padé approach to obtain sufficient efficiency. As a result, the improvement has been verified by solving two cases beside a mean value comparison of the homotopy analysis method's squared error with the improved form.

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1. INTRODUCTION

During the past few decades, the fractional calculus gain much consideration because of its applicability in a lot of science and engineering fields like: electrical networks, phenomena in the fluid flow areas, probabilistic and statistics based decisions, chemical-physics, electrochemistry, and signal processing. That generally modelled by linear/nonlinear fractional differential equations [1]–[3]. To solve equations of different categories like linear or nonlinear, ordinary differential or partial differential equations, integer or fractional a number of methods have been used, for example adomian's decomposition method [4]–[6], homotopy perturbation method [7], he's variational iteration method [8], homotopy analysis method [9].

Among a multitude of available techniques to tackle nonlinear equations, homotopy analysis method (HAM) has gained tremendous popularity. Initially, Liao proposed in his doctoral thesis [10]–[13] the main method to develop the basic concepts of homotopy analysis in topical geometry to propose a general method of analysis for solving nonlinear problems. The homotopy analysis method does not depend on a small parameter, unlike other analytical techniques such as the method of decomposition, the homotopy perturbation method (HPM) as these methods are particular cases of the homotopy analysis method and which gives us a sufficient way to control and modify the convergence of solution chain by controlling the value of error by means of the convergence control parameter h. In this chapter, we have used homotopy analysis method with Padé approximations to solve linear partial differential equations of fractional order [14]–[16], the general formula of which is:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + A(x)\frac{\partial u}{\partial x} + B(x)\frac{\partial^2 u}{\partial x^2} + C(x)u = h(x,t) \quad (1)$$

where

$$(x,t) \in [0,1] \times [0,T], n-1 < \alpha \leq n, n \epsilon N \quad (2)$$

and initial conditions

$$\frac{\partial^k u}{\partial t^k}(x,0) = f_k(x) \quad k = 0,1,\dots,m-1 \quad (3)$$

2. DEFINITION

Let $f(x), x > 0$ be a real function which is in the space CM, $M \epsilon R$ where the real number $k > M$, for $f(x) = x\, kg(x)$ where $g(x) \in [0,\infty)$ which is in the space CmM if and only if $f(x)^m \in C_M, m \in N$ [17].

2.1. Riemann-Liouville fractional integer

The Riemann-Liouville fractional integral operator of order of a function $f(x) \in C_\mu, \mu \geq -1$ defined as in (4) and (5) [18]:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt\,, x > 0 \quad (4)$$

$$J^0 f\,(x) = f(x) \quad (5)$$

for $(x) \in C\mu, \mu \geq -1, \alpha, \beta \geq 0, \gamma \geq -1$ properties of the operator $J\alpha$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \quad (6)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \quad (7)$$

2.2. Devrative Caputo fractional

The fractional derivative of (x) [19] in the Caputo sense is defined as (8) [20]–[22]:

$$D_x^\alpha f(x) = J^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt \quad (8)$$

for $(x) \in Cn, \mu \geq -1, \alpha, \beta \geq 0, \gamma \geq -1, n-1 < \alpha \leq n, n \in \mathbb{N}$ properties of the operator $D\alpha$

$$D_x^\alpha D_x^\beta f(x) = D_x^{\alpha+\beta} f(x) = D_x^\beta D_x^\alpha f(x) \quad (9)$$

$$D_x^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, x > 0 \quad (10)$$

3. METHOD

Let's get the following nonlinear differential equation that is formed [23]–[25]: where N (nonlinear operator) (unknown function) (the independent variable). Let y_0 be a first guess of the analytical solution y, $h \neq 0$ an assistant parameter, $H(t) \neq 0 \;\forall\; t \in R$ an assistant function and L an assistant linear operator with the property that $L[y(t)] = 0$ when $y(t) = 0$. Then using $q \in [0,1]$ as parameter for embedding. We construct a homotopy that's called zero-order deformation:

$$N[y(t)] = 0 \quad for\; t \geq 0 \quad (11)$$

Having considerable selection of the initial guess y_0 should be stressed, the assistant linear operator. Here is L, and nonzero assistant function H(t). When q=0, can see the deformation equation zero-order (12) becomes

$$(1-q)\, L[\emptyset(t\,,q\,) - y_0(t)] = qhH(t)N[\emptyset(\,t\,,q\,)] \quad (12)$$

and when q=1, "since h 0 and H(t) /= 0", the zero-order deformation (12) is tantamount to

$$\emptyset(t,0) = y_0(t) \quad (13)$$

and when $q = 1$, since $h \neq 0$ and $H(t) \neq 0$, the zero-order deformation (12) is equivalent to

$$\emptyset(t,1) = y(t) \quad (14)$$

Thus, based on (13) and (14), the embedding-parameter q rises from 0 to 1, $\emptyset(t,q)$ Different continually from $y_0(t)$ to the same solution $y(t)$. This type of continous variation is deformation in homotopy.

Also, Taylor's theorem provides, $\emptyset(t,q)$ might deal with a power series of q in the same expand:

$$\emptyset(t,q) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t)\, q^m \quad (15)$$

where

$$y_m(t) = \frac{1}{m!}\frac{\partial^m \emptyset(t,q)}{\partial q^m}\Big|_{q=0} \quad (16)$$

If y_0, L, h and the power series (9) of $\phi(t,q)$ converge at $q = 1$, then we have the solution series under those assumptions:

$$y(t) = \emptyset(t,1) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t) \quad (17)$$

for brevity, define the vector:

$$\vec{y}_n(t) = \{y_0(t), y_1(t), y_2(t), \dots, y_n(t)\} \quad (18)$$

According to (17), the governed-equation of $y_m(t)$, also derived using the zero-order as in the (12) taht accour by the zero order deformation differentiation in (12) m -times with respective to q later to be divided by m! and later to set $q = 0$, to get m th -order deformation:

$$L[y_m(t) - \chi_m y_{m-1}(t)] = hH(t)\, R_{m-1}\left(\vec{y}_{m-1}(t)\right) \quad (19)$$

where:

$$R_{m-1}\left(\vec{y}_{m-1}(t)\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\emptyset(t,q)]}{\partial q^{m-1}}\Big|_{q=0} \quad (20)$$

and

$$\chi_m = \{0, \quad m \leq 1 \quad 1, \quad m > 1 \quad (21)$$

4. PROPOSED ALGORITHM

The algorithm was rewritten the homotopy analysis method and then linked to the pseudoPadé approximations. To find the initial evalution u0(x, t), we integrate (1) by Riemann-Liouville integration method with respect to the variable t and substituting the initial conditions to obtain

$$u(x,t) = \sum_{k=0}^{n-1} u^{(k)}(x,0^+)\frac{t^k}{k!} + J_t^\alpha\left(A(x)\frac{\partial u}{\partial x} + B(x)\frac{\partial^2 u}{\partial x^2} + C(x)u\right) + J^\alpha h(x,t) \quad (22)$$

By omitting the term JαA(x)∂u +B(x) ∂2u+C(x)u in the right side in (22), we obtain the initial evolution u0(x, t) by the formula

$$u_0(x,t) = \sum_{k=0}^{n-1} u^{(k)}(x,0^+)\frac{t^k}{k!} + J^\alpha h(x,t) \quad (23)$$

By using the deformation (19) of order m with assuming $L = D\alpha$, $H(t) = 1$ where n ∈ N and n -1 < α ≤ n, we obtain the following sequential formula:

$$uD^\alpha u_m(x,t) = \chi_m\, D^\alpha\, u_{m-1}(x,t) + h\, R_{m-1}\left(u_{m-1}(x,t)\right) \quad (24)$$

$$R_{m-1}\left(u_{m-1}(x,t)\right)=\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\Big(D^{\alpha}\emptyset(x,t,q)+A(x)\frac{\partial u(\emptyset(x,t,q))}{\partial x}+B(x)\frac{\partial^{2}u(\emptyset(x,t,q))}{\partial x^{2}}+ C(x)u\big(\emptyset(x,t,q)\big)-h(x,t)\Big)\Big|_{q=0} \quad (25)$$

To find approximate iterations $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$, we apply Jα Riemann-Liouville inte- gration on The both side of (24) to obtain

$$u_m(x,t)=\chi_m u_{m-1}(x,t)-\chi_m\sum_{j=0}^{n-1}u_{m-1}{}^{(j)}(x,0^{+})\frac{t^{j}}{j!}+h\,J_t^{\alpha}\left[R_{m-1}\left(u_{m-1}(x,t)\right)\right] \quad (26)$$

By substituting values of m=1, 2, 3, … in (26), we obtain the approximated solutions $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$. By substituting the approximated iterations $u_1(x,t), u_2(x,t), u_3(x,t)$,with $u_0(x,t)$ in the solution series $\emptyset_m(x,t)$, we get:

$$\emptyset(x,t)=\sum^{m-1}us\,(x,t) \quad (27)$$

Then, substituting the values of h and α, we obtain an approximated solution for initial values problems (27) given by the form:

$$\emptyset_m(x,t)=\sum_{r=0}^{k}a_r t^{\beta_r} \quad (28)$$

where a_0, a_1,…, a_k are constants, β_0, β_1,…, β_k are different positive powers, m is the number of iterations and k is specified according to the given example.

Now, we connect the series (28) by Padé approximation using the following assumption:

$$u(x,t)=\sum_{k=0}^{n-1}u^{(k)}(x,0^{+})\frac{t^{k}}{k!}+J_t^{\alpha}\left(A(x)\frac{\partial u}{\partial x}+B(x)\frac{\partial^{2}u}{\partial x^{2}}+C(x)u\right)+J^{\alpha}h(x,t)$$
$$t^{\omega}=z \quad (29)$$

where ω is fraction number, after substituting in the approximated solution (28), we obtain a series that has the form

$$\emptyset_m^{*}(x,z)=\sum_{r=0}^{k}a_r x^{v_r} \quad (30)$$

where v_0, v_1, …, v_k are positive powers. By using Padé approximation of order M and N where M, N ∈ N ∪ {0}, we can convert the sequence (30) to a fractional series:

$$R_{N,M}(x,z)=\frac{P_N(x,z)}{Q_M(x,z)} \quad (31)$$

Finally, by substituting the transformation $z=t^{\omega}$in the series (31), we obtain the Padé series with fractional powers.

5. NUMERICAL EXAMPLES

Suppose we have the following one dimensional heat equation that has a fractional order [4], [19], [20].

$$D_t^{\alpha}u=\frac{1}{\pi^{2}}u_{xx}\,,0\;<\;\alpha\leq\;1\,for\;t>0 \quad (32)$$

With condition (x, 0)=sin πx, The exact solution for the (32) is

$$u(x,t)=\;sin\;\pi x\sum_{K=0}^{\infty}\frac{(-t^{\alpha})^{K}}{\Gamma(\alpha K+1)} \quad (33)$$

The solution: depending on the algorithm mentioned in the fifth paragraph, we get approximations:

$$u_0(x,t) = sin\ \pi x$$

$$u_1(x,t) = \frac{2xht^{\alpha}}{\Gamma(\alpha\ +\ 1)} + \frac{2ht^{2\alpha}}{\Gamma(2\alpha\ +\ 1)} + \frac{2ht^{\alpha}}{\Gamma(\alpha\ +\ 1)}$$

$$u_2(x,t) = (1+2h)\frac{ht^{\alpha}sin\pi x}{\Gamma(\alpha+1)} + \frac{h^2t^{2\alpha}cos\pi x}{\Gamma(2\alpha+1)}$$

We repeat this process to obtain $u_4(x,t)$. By adding approximated iterations $u_0(x,t)$, $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$, $u_4(x,t)$, we obtain an approximated solution Ø5(x, t) where:

$$\emptyset_5(x,t) = \sum_{m=0}^{4} u_m(x,t) = sin\pi x + \frac{ht^{\alpha}sin\pi x}{\Gamma(\alpha+1)}(3+5h+2h^2) + \frac{h^2t^{2\alpha}cos\pi x}{\Gamma(2\alpha+1)}(2+h) + +(1+2h)\frac{h^2t^{2\alpha}sin\pi x}{\Gamma(2\alpha+1)} + (1+h)\frac{h^3t^{3\alpha}cos\pi x}{\Gamma(3\alpha+1)} \quad (34)$$

Now, by taking different values of α with the best value of h at each value of α in the solution series (34) and then relating them with Padé approximation:

- When α=0.25 and h=–0.18 and assuming that $t^{0.25} = z$, then calculating [1/2] the Padé's approximations, and remembering that z =$t^{0.25}$, we get:

$$[1/2] = \frac{sin(3.1415x) - 1.1238t^{0.25}sin(3.1415x)}{1 - .5.1442e^{-1}t^{0.25} + 1.7940t^{0.5}} \quad (35)$$

- When α=0.5 and h=0.25 and assuming that $t^{0.5} = z$ in the solution seriesØ 5(x, t), then calculating [1/2] the Padé approximations, and remembering that z=$t^{0.5}$, we get:

$$[1/2] = \frac{sin(3.1415x) - 1.4094t^{0.5sin}(3.1415x)}{1 - 6.3806_t{}^{0.5} + 2.2221t} \quad (36)$$

- When α=0.75 and h=0.22 and Assuming that $t^{0.75} = z$ in the solution series Ø5(x, t),then calculating [1/2] the Padé's approximations, and remembering that $z\ = t^{0.75}$, we get:

$$[1/2] = \frac{sin(3.1415x) - 1.2292t^{0.75}sin(3.1415x)}{1 - 5.4392e^{-1}{}_t{}^{0.75} + 1.3873t^{1.5}} \quad (37)$$

Suppose we have a one dimensional linear no homogenous Burgers equation of fractional order [26].

$$D_t^{\alpha}u + u_x - u_{xx} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2 \quad (38)$$

when:

$$0 < \alpha \leq 1, t\ 0, x \in R \text{ With condition } u(x,0) = x^2 \quad (39)$$

the exact solution for the problem (38) is

$$u(x,t) = x^2 + t^2 \quad (40)$$

The solution: depending on the algorithm mentioned in the fifth paragraph, we get approximations:

$$u_0(x,t) = x^2 + t^2 + \frac{t^{\alpha}}{\Gamma(\alpha+1)}(2x-2\ )$$

$$u_1(x,t) = \frac{2xht^{\alpha}}{\Gamma(\alpha+1)} + \frac{2ht^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2ht^{\alpha}}{\Gamma(\alpha+1)}$$

$$u_2(x,t) = (1+2h+h^2)\frac{2xht^{\alpha}}{\Gamma(\alpha+1)} - \frac{2ht^{\alpha}}{\Gamma(\alpha+1)}(1+h+h^2) + \frac{2ht^{2\alpha}}{\Gamma(2\alpha+1)}(1+4h+3h^2)$$

We repeat this process to obtain $u_4(x,t)$. By adding approximated iterations $u_0(x,t)$, $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$, $u_4(x,t)$, we obtain an approximated solution Ø5(x, t) where:

$$\emptyset_5(x,t)=\sum_{m=0}^{4} u_m(x,t)=x^2+t^2+\frac{t^\alpha}{\Gamma(\alpha+1)}(2x-2)+(3+5h+2h^2)\frac{2xht^\alpha}{\Gamma(\alpha+1)}-\frac{2ht^\alpha}{\Gamma(\alpha+1)}(3+3h+2h^2)+\frac{2ht^{2\alpha}}{\Gamma(2\alpha+1)}(3+10h+8h^2+h^3)+ \quad (41)$$

Now, by taking different values of α with the best value of h at each value of α in the solution series (41) and then relating them with Padé approximation:

- When α=0.25 and h=−0.7 and assuming that $t^{0.25}=z$ in the solution series Ø5(x, t), then calculating the PA [5/6] the Padé's approximations, and remembering that $z=t^{0.25}$, we get:

$$PA\frac{5}{6}=\frac{x2+e^{-29}(1.7872e22x\ -1.7872e27)t0.25\ -1.7061e-1t2(42)}{(1+(34.4190x2-1.2864e-1x+6.4321e-2)t+(6.1401e-1-6.1401e-1)t1.25+5.8612t15)}$$

$$+\frac{e^{-29}(3.4419e^{30}x^4\ 1.2864e^{28}e^3+6.4323e^{27}x^2t+e^{-29}(1.2291e^{29}x^3\ 1.2314e^{29}x^2 3.4489))}{(:1+W(3h4e.4n190x^2-1.2864e^1x+6.4321e^{-2})t+(6.1401e^{-1}-6.1401e^{-1})t^{1.25}+5.8612t^{15})} \quad (42)$$

- When α=0.5 and h=0.85 and assuming that $t^{0.5}=z$ in the solution series 5(x, t),Ø then calculating [1/2] the Padé's approximations, and remembering that $z=t^{0.5}$, we get:

$$PA\left[\frac{1}{2}\right]=\frac{\left(\frac{3.5861e^{54}x^4+4.0789e^{50}x^3+2.0394e^{50}x^2}{8.1939e^{51}x^3-8.1944e^{51}x^2+6.9908e^{47}x-2.3300+2.0394}\right)}{\left(\left(1+\frac{4.0968e^{51}x-4.0968e^{51}}{3.5861e^{54}x^2-4.0789e^{50}+2.0394e^{50}}\right)t^{0.5}+\frac{8.2297e^{52}t}{3.5861e^{54}x^2-4.0789e^{50}+2.0394e^{50}}\right)} \quad (43)$$

- When α=0.75 and h=0.6 and Assuming that $t^{0.75}=z$ in the solution series Ø5(x, t), then calculating [2/3] the Padé's approximations, and remembering that $z=t^{0.75}$, we get:

$$PA[2/3]=\frac{x^2}{1+\frac{3.2221e^{-56}(-1.7289e^{54}x+1.7289e^{54})t^{0.75}}{x^2}} \quad (44)$$

6. RESULTS

The results of this paper show that the proposed homotopy method analysis solve partial differential equations efficiently. The results have been verified with other works to show that the proposed method has a better mean square error and exact solution. The results compared with both HAM and HAM-PA when t=0.01 and 0.1 as shown in Tables 1 to 5.

Table 1. The obtained results which compared to the exact solution for instance example 5.1, at t=0.01

x	Exact Solution	HAM	HAM-PA
0	0	0	0
0.1	$2.5779e-1$	$2.5518e-1$	$2.5639e-1$
0.2	$4.9035e-1$	$4.8538e-1$	$4.8769e-1$
0.3	$6.7491e-1$	$6.6807e-1$	$6.7124e-1$
0.4	$7.9340e-1$	$7.8537e-1$	$7.8910e-1$
0.5	$8.3423e-1$	$8.2578e-1$	$8.2970e-1$
0.6	$7.9340e-1$	$7.8537e-1$	$7.8910e-1$
0.7	$6.7491e-1$	$6.6807e-1$	$6.7124e-1$
0.8	$4.9035e-1$	$4.8538e-1$	$4.8769e-1$
0.9	$2.5779e-1$	$2.5518e-1$	$2.5639e-1$
1.0	$-4.1471e-31$	$-4.1051e-1$	$-4.1246e-1$

Table 2. The obtained results which compared to the exact solution for instance example 5.2, at t=0.01

x	Exact Solution	HAM	HAM-PA
0	e^{-4}	$-2.2612e^{-2}$	$-5.6468e^{-3}$
0.1	$1.01e^{-2}$	$-1.2047e^{-2}$	$4.8589e^{-3}$
0.2	$4.01e^{-2}$	$1.851e^{-2}$	$3.5250e^{-2}$
0.3	$9.01e^{-2}$	$6.9082e^{-2}$	$8.5535e^{-2}$
0.4	$1.601e^{-1}$	$1.3964e^{-1}$	$1.5572e^{-1}$
0.5	$2.501e^{-1}$	$2.3021e^{-1}$	$2.4582e^{-1}$
0.6	$3.601e^{-1}$	$3.4077e^{-1}$	$3.5586e^{-1}$
0.7	$4.901e^{-1}$	$4.7134e^{-1}$	$4.8585e^{-1}$
0.8	$6.401e^{-1}$	$6.2190e^{-1}$	$6.3579e^{-1}$
0.9	$8.101e^{-1}$	$7.9247e^{-1}$	$8.0572e^{-1}$
1.0	1.0001	$9.8303e^{-1}$	$9.9564e^{-1}$

Table 3. Comparison of the mean square error for each of the HAM and HAM-PA [5/6]

Algorithm Name	Error
MSE- HAM	3.9868e − 4
MSE - HAM-PA	2.1430e − 5

Table 4. Comparison of the mean square error for each of the HAM and HAM-PA [1/2]

Algorithm Name	Error
MSE-HAM	8.3458e − 8
MSE - HAM-PA	7.7009e − 8

Table 5. Comparison of the mean square error for each of the HAM and HAM-PA [2/3]

Algorithm Name	Error
MSE-HAM	$1.5467e-6$
MSE - HAM-PA	$8.7674e-7$

7. CONCLUSION

This study, the efficiency of the HAM with Padé approximations to solve partial fractional linear questions has been proven. The results showed the possibility of Padé approximations to improve the results of the HAM. Padé approximation only solves sequences with natural bases. Therefore, we used the hypothesis tω=x where ω is the number of fractional basis to normal basis.

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