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A new contraction based on $\mathscr{H}$-simulation functions in the frame of extended b-metric spaces and application

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ABSTRACT

The conceptions of b-metric spaces and metric spaces play a remarkable role in proving many theorems of uniqueness and existence solution of such equations as integral or differential equations. The conception of extended b-metric spaces is considered as a generalization concept of b-metric spaces and metric spaces and this concept was employed to unify some new fixed point results in the literature. On the other hand, a new concept of Simulation functions founded in 2020 in the name of $\mathscr{H}$-simulation functions and employed this class of functions to unify some fixed point results in the literature. The main objective of this manuscript, is to establish a new contraction namely, $(\gamma, \phi, \theta)_{\mathscr{H}}$-contraction, this contraction based on the concepts of extended b-metric spaces and the class of functions ($\mathscr{H}$-simulation functions) and the class of functions Θ-functions and the class of functions Φ-functions, We utilize this new contraction to unify the uniqueness and existence of fixed point results in the literature. In addition, some illustrative examples and an interesting application were established to show the novelty of our work.

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1. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

The study of fixed point theory has taken a wide range in analysis and applied mathematics since Banach's result [1] which considered the outstanding result in this field. The theorem of Banach asserts the existence and uniqueness of fixed point for any contraction mapping on a complete metric space. Then after, many researchers generalized the result of Banach in two directions; some of them by replacing the frame of distance space (for example see [2]–[16]), and the others by improving the contraction condition (for example see [17]–[30]). In this manuscript, we consider the following notations: $\mathscr{W}$ is a non empty set, $\mathbb{R}$ the set of all real numbers, $\mathbb{N}$ the set of all natural numbers and $\mathscr{G}$ the set of all fixed point for a self mapping $g : \mathscr{W} \to \mathscr{W}$.

The notion of extended b-metric spaces was established by Kamran *et al.* [31] as:

Definition 1 [31]: On $\mathscr{W}$, consider the function $\gamma : \mathscr{W} \times \mathscr{W} \to [1, +\infty)$, the mapping $E_\gamma : \mathscr{W} \times \mathscr{W} \to [0, +\infty)$ is said to be an extended b-metric space if the following conditions hold:

- $[(E_\gamma 1)]$ $E_\gamma(w', w) = 0$ iff $w' = w$,
- $[(E_\gamma 2)]$ $E_\gamma(w', w) = E_\gamma(w, w')$,
- $[(E_\gamma 3)]$ $E_\gamma(w', w) \leq \gamma(w', w)\big[E_\gamma(w', w'') + E_\gamma(w'', w)\big]$ $\forall\, w'', w', w \in \mathscr{W}$.

From now on, $(\mathscr{W}, E_\gamma)$ is referred as extended b-metric space.

Remark 1: It is clear that if $\gamma(w,w') = s \geq 1$ in $(\mathscr{W}, E_\gamma)$, then the space will be b-metric space.
Definition 2 [31]: On $\mathscr{W}$, consider $(\mathscr{W}, E_\gamma)$ and a sequence (w_r) in $\mathscr{W}$. Then:
(w_r) converges to an element $w' \in \mathscr{W}$ if

$$\lim_{r\to+\infty} E_\gamma(w_r, w') = 0.$$

(w_r) is Cauchy if

$$\lim_{r,s\to+\infty} E_\gamma(w_r, w_s) = 0.$$

Definition 3: On $(\mathscr{W}, E_\gamma)$ we say that the function $\gamma : \mathscr{W} \times \mathscr{W} \to [1, +\infty)$ is bounded if there is an integer Γ such that for all $w_1, w_2 \in \mathscr{W}$ we have $\gamma(w_1, w_2) \leq \Gamma$.

Pioneer mathematicians, namely Khojasteh *et al.* [32] established the notion of simulation functions in 2017 and they used them to unify many fixed point results in the literature. And then, many others mathematicians established other types of simulation functions such as Cho [33] established $\mathscr{L}$-simulation functions and Bataihah *et al.* [34] established the notion of $\mathscr{H}$-simulation functions.
Definition 4 [34]: A function $h : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$ is said to be $\mathscr{H}$-simulation function if $h(w, w^{'}) \leq \frac{w^{'}}{w}$, for all $w, w^{'} \in [1, +\infty)$.

We referred by $\mathscr{H}$ the class of all $\mathscr{H}$-simulation functions.
Remark 2 [34]: If $h \in \mathscr{H}$, then for all sequences $(w_r), (w_r^{'})$ in $[1, +\infty)$, $1 \leq \lim_{r\to+\infty} w_r^{'} < \lim_{r\to+\infty} w_r$ implies $\limsup_{r\to+\infty} d(w_r, w_r^{'}) < 1$.
Definition 5 [34], [35]: Suppose Θ denotes the class of all non-decreasing functions $\theta : [0, +\infty) \to [1, +\infty)$ that satisfying: (1) (Θ_1) θ is continuous on $[0, +\infty)$ and (2) (Θ_2) For each sequence $\{w_r^{'}\}$ in $[0, +\infty)$, $\lim_{r\to+\infty} \theta(w_r^{'}) = 1$ iff $\lim_{r\to+\infty} w_r = 0$.
Remark 3: Suppose $\theta \in \Theta$. Then $\theta^{-1}(\{1\}) = 0$.
Example 1 [34]: The following functions $h : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$ are $\mathscr{H}$-simulation functions

- $h(w_1, w_2) = \dfrac{\min\{w_1, w_2\}}{\max\{w_1, w_2\}}$
- $h(w_1, w_2) = \dfrac{w_2}{w_1 + |\ln(\frac{w_2}{w_1})|}$
- $h(w_1, w_2) = \dfrac{w_2}{w_1 + \sqrt{w_2}}$
- $h(w_1, w_2) = \dfrac{w_2^2}{1 + w_1 w_2}$

Definition 6 [36]: Suppose Φ denotes the class of all non-decreasing functions $\phi : [1, +\infty) \to [1, +\infty)$ that satisfying: (1) (Φ_1) ϕ is and continuous on $[1, +\infty)$ and (2) (Φ_2) $\forall\, w^{'} > 1$, $\lim_{r\to+\infty} \phi^r(w^{'}) = 1$.
Remark 4 [36]: Suppose that $\phi \in \Phi$. Then $\phi(1) = 1$ and $\phi(w^{'}) < w^{'}$ $for\ all$ $1 < w^{'}$.

2. MAIN RESULTS

Definition 7: Suppose there is $(\mathscr{W}, E_\gamma)$. A self mapping g on $\mathscr{W}$ is called $(\gamma, \phi, \theta)_{\mathscr{H}}$ if there are $h \in \mathscr{H}$, $\theta \in \Theta$, $\phi \in \Phi$ and $\lambda \in (0, 1)$ such that for all $w_1, w_2 \in \mathscr{W}$ we have (1).

$$\gamma(w_1, w_2) \leq h\left(\theta E_\gamma(gw_1, gw_2), \phi\theta\lambda E_\gamma(w_1, w_2)\right). \tag{1}$$

Lemma 1: Suppose that g satisfies the condition of $(\gamma, \phi, \theta)_{\mathscr{H}}$-contraction. Then for all $w_1, w_2 \in \mathscr{W}$, we have the following results:

- $E_\gamma(w_1, w_2) = 0$ implies $E_\gamma(gw_1, gw_2) = 0$,
- $0 < E_\gamma(w_1, w_2)$ implies $E_\gamma(gw_1, gw_2) < \dfrac{\lambda}{\gamma(w_1, w_2)} E_\gamma(w_1, w_2)$.

Proof 1: Suppose $E_\gamma(w_1, w_2) = 0$. Then, by condition 1, we have

$$1 \leq \gamma(w_1, w_2) \leq \theta E_\gamma(gw_1, gw_2) \leq \phi\theta\lambda E_\gamma(w_1, w_2) = 1.$$

Hence the result, suppose $0 < E_\gamma(w_1, w_2)$. Then, condition 1 implies

$$\begin{aligned}\gamma(w_1, w_2) &\leq h(\theta E_\gamma(gw_1, gw_2), \phi\theta\lambda E_\gamma(w_1, w_2)) \\ &\leq \frac{\phi\theta\lambda E_\gamma(w_1, w_2)}{\theta E_\gamma(gw_1, gw_2)} \\ &< \frac{\theta\lambda E_\gamma(w_1, w_2)}{\theta E_\gamma(gw_1, gw_2)}.\end{aligned}$$

So, $\gamma(w_1, w_2)\theta E_\gamma(gw_1, gw_2)) < \theta\lambda E_\gamma(w_1, w_2)$. and so, we have $\theta E_\gamma(gw_1, gw_2)) < \frac{1}{\gamma(w_1, w_2)}\theta\lambda E_\gamma(w_1, w_2)$.
Since θ is non-decreasing function hence the result.
Lemma 2: Suppose there is $(\mathscr{W}, E_\gamma)$ and a self mapping g on $\mathscr{W}$ is a $(\gamma, \phi, \theta)_{\mathscr{H}}$-contraction. Then $\mathscr{G}$ consists of at most one element.
Proof 2: Assume to the contrary that, there are two elements $w, w^{'} \in \mathscr{G}$, then $0 < E_\gamma(w, w^{'})$, using lemma 1, we get that

$$E_\gamma(w, w^{'}) = E_\gamma(gw, gw^{'}) < \frac{\lambda}{\gamma(w, w^{'})}E_\gamma(w, w^{'}) \leq \lambda E_\gamma(w, w^{'}) < E_\gamma(w, w^{'}).$$

A contradiction and so $E_\gamma(w, w^{'}) = 0$. Hence, $w = w^{'}$.

In $(\mathscr{W}, E_\gamma)$, let us start with $w_0 \in \mathscr{W}$ and $g : \mathscr{W} \to \mathscr{W}$ is a self mapping, then the sequence (w_r) where $w_r = gw_{r-1},\ r \in \mathbb{N}$ is called the Picard sequence generated by g at w_0.
Lemma 3: Suppose there is $(\mathscr{W}, E_\gamma)$. Let $g : \mathscr{W} \to \mathscr{W}$ be an $(\gamma, \phi, \theta)_{\mathscr{H}}$-contraction. Then,

$$\lim_{r\to\infty} E_\gamma(w_r, w_{r+1}) = 0 \tag{2}$$

for any initial point $w_0 \in \mathscr{W}$ where (w_r) is the Picard sequence generated by g at w_0.
Proof 3: Suppose that $w_0 \in \mathscr{W}$ be any initial point and (w_r) is the Picard sequence generated by g at w_0. If there is $K \in \mathbb{N}$ such that $E_\gamma(w_{K+1}, w_K) = 0$, then by lemma 1, we get that $E_\gamma(w_{r+1}, w_r) = 0$ for all $r \geq K$, hence the result.

Now, suppose that $0 < E_\gamma(w_{r+1}, w_r)$ for all $r \in \mathbb{N}$. By lemma 1, we get

$$E_\gamma(w_{r+1}, w_r) < \frac{\lambda}{\gamma(w_r, w_{r-1})}E_\gamma(w_r, w_{r-1}) < \lambda E_\gamma(w_r, w_{r-1}).$$

Consequently, the sequence $\{E_\gamma(w_{r+1}, w_r) : r \in \mathbb{N}\}$ is a non increasing sequence in $[0, +\infty)$, thus, there is $\alpha_0 \geq 0$ such that $\lim_{r\to+\infty} E_\gamma(w_{r+1}, w_r) = \alpha_0$. We claim that $\alpha_0 = 0$. Suppose to the contrary that, $\alpha_0 > 0$. Let $\alpha_r = \theta E_\gamma(w_{r+1}, w_r)$ and $\beta_r = \phi\theta\lambda E_\gamma(w_r, w_{r-1})$. Then $1 \leq \lim_{r\to+\infty}\gamma(w_r, w_{r-1}) \leq \lim_{r\to+\infty}\beta_r < \lim_{r\to+\infty}\alpha_r$ by 1 and remark 2, we have

$$1 \leq \limsup_{r\to+\infty} h(\theta E_\gamma(w_{r+1}, w_r), \phi\theta\lambda E_\gamma(w_r, w_{r-1})) < 1,$$

a contradiction, and so, $\lim_{r\to+\infty} E_\gamma(w_{r+1}, w_r) = 0$.

Theorem 4: Suppose $(\mathscr{W}, E_\gamma)$ is complete where γ is bounded by $\frac{1}{\lambda}$. Suppose there are $h \in \mathscr{H}$, $\theta \in \Theta$, $\phi \in \Phi$ and $\lambda \in (0, 1)$ such that $g : \mathscr{W} \to \mathscr{W}$ is a $(\gamma, \phi, \theta)_{\mathscr{H}}$-contraction. Then, there is a unique element in $\mathscr{G}$. Moreover, the sequence (w_r), where $w_{r+1} = gw_r$, $r \geq 0$ converges for any $w_0 \in \mathscr{W}$ and $\lim_{r\to\infty} w_r \in \mathscr{G}$.
Proof 4: Let us start with $w_0 \in \mathscr{W}$ and the Picard sequence (w_r) in $\mathscr{W}$ which generated by g at w_0. If there is

$t \in \mathbb{N}$ such that $E_\gamma(w_{t+1}, w_t) = 0$, then $w_t \in \mathscr{G}$. So, we may assume that, for each $r \in \mathbb{N}$, $E_\gamma(w_{r+1}, w_r) \neq 0$. Thus, by lemma 3, we get $\lim_{r\to+\infty} E_\gamma(w_{r+1}, w_r) = 0$.

Next, we want to show that, (w_r) is a Cauchy sequence, i.e., $\lim_{r,t\to+\infty} E_\gamma(w_r, w_t) = 0$. Assume to the contrary that, $\lim_{r,t\to+\infty} E_\gamma(w_r, w_t) \neq 0$. Therefore, there exist $\epsilon > 0$ and two sub-sequences (w_{r_k}) and (w_{t_k}) of (w_r) such that (t_k) is chosen as the smallest index for which (3),

$$\epsilon \leq E_\gamma(w_{r_k}, w_{t_k}), t_k > r_k > k, \tag{3}$$

which implies that (4).

$$E_\gamma(w_{r_k}, w_{t_k-1}) < \epsilon. \tag{4}$$

We claim that, $\lim_{k\to+\infty} E_\gamma(w_{r_k-1}, w_{t_k}) = \epsilon$ and $\lim_{k\to+\infty} E_\gamma(w_{r_k}, w_{t_k+1}) = \lambda\epsilon$. To prove our claim, set $\alpha_k = E_\gamma(w_{r_k-1}, w_{t_k})$. Then, by utilizing lemma 1, 3, 4 and $E_\gamma 3$ of the definition of E_γ, we have

$$\begin{aligned}\epsilon \leq E_\gamma(w_{r_k}, w_{t_k}) &\leq \frac{\lambda}{\gamma(w_{r_k-1}, w_{t_k-1})} E_\gamma(w_{r_k-1}, w_{t_k-1}) \\ &\leq \frac{\lambda}{\gamma(w_{r_k-1}, w_{t_k-1})} \gamma(w_{r_k-1}, w_{t_k-1})[E_\gamma(w_{r_k-1}, w_{t_k}) + E_\gamma(w_{t_k}, w_{t_k-1})] \\ &= \lambda[E_\gamma(w_{r_k-1}, w_{t_k}) + E_\gamma(w_{t_k}, w_{t_k-1})].\end{aligned}$$

By taking the limit inferior as $k \to +\infty$ and by taking into consideration (2), we have (5),

$$\epsilon \leq \liminf_{k\to+\infty} \alpha_k. \tag{5}$$

also,

$$\begin{aligned}E_\gamma(w_{r_k-1}, w_{t_k}) &\leq \frac{\lambda}{\gamma(w_{r_k-2}, w_{t_k-1})} E_\gamma(w_{r_k-2}, w_{t_k-1}) \\ &\leq [\lambda E_\gamma(w_{r_k-2}, w_{r_k}) + \lambda E_\gamma(w_{r_k}, w_{t_k-1})] \\ &< \lambda\gamma(w_{r_k-2}, w_{r_k})[(E_\gamma(w_{r_k-2}, w_{r_k-1}) + E_\gamma(w_{r_k-1}, w_{r_k})] + \lambda\epsilon.\end{aligned}$$

By taking the limit superior as $k \to +\infty$ and by taking into consideration (2), we have (6),

$$\limsup_{k\to+\infty} \alpha_k \leq \epsilon. \tag{6}$$

Consequently,

$$\lim_{k\to+\infty} \alpha_k = \epsilon. \tag{7}$$

Next, let $\beta_k = E_\gamma(w_{r_k}, w_{t_k+1})$. By lemma 1 we have

$$E_\gamma(w_{r_k}, w_{t_k+1}) \leq \frac{\lambda}{\gamma(w_{r_k-1}, w_{t_k})} E_\gamma(w_{r_k-1}, w_{t_k}) \leq \lambda E_\gamma(w_{r_k-1}, w_{t_k}).$$

For both sides, take the limit superior, we have (8),

$$\limsup_{k\to+\infty} \beta_k \leq \lambda\epsilon. \tag{8}$$

Also, we get

$$\begin{aligned}\epsilon \leq E_\gamma(w_{r_k}, w_{t_k}) &\leq \gamma(w_{r_k}, w_{t_k})[E_\gamma(w_{r_k}, w_{t_k+1}) + E_\gamma(w_{t_k+1}, w_{t_k})] \\ &\leq \frac{1}{\lambda}[E_\gamma(w_{r_k}, w_{t_k+1}) + E_\gamma(w_{t_k+1}, w_{t_k})]\end{aligned}$$

For both sides, take the limit inferior to get (9),

$$\lambda\epsilon \leq \liminf_{k\to+\infty} \beta_k. \tag{9}$$

Consequently, we have (10).

$$\lim_{k\to+\infty} \beta_k = \lambda\epsilon. \tag{10}$$

By employing the properties of θ and ϕ, we get

$$\phi(\theta(\lambda\epsilon)) < \theta(\lambda\epsilon).$$

Now, by letting $c_k = \theta(\beta_k)$ and $d_k = \phi(\theta(\lambda\alpha_k))$, then $\lim_{k\to+\infty} c_k > \lim_{k\to+\infty} d_k \geq 1$. So, remark 2 and condition 1 yield that

$$1 \leq \limsup_{k\to+\infty} h(c_k, d_k) < 1.$$

A contradiction. Thus, $\lim_{r,t\to+\infty} E_\gamma(w_r, w_t) = 0$. Hence, (w_r) is a Cauchy sequence. So, there is an element $\omega' \in \mathscr{W}$ such that $\lim_{r\to+\infty} w_r = \omega'$. To show $\lim_{r\to+\infty} g\omega' = \omega'$.

$$\begin{aligned}\gamma(\omega', w_r) &\leq h(\theta E_\gamma(g\omega', gw_r), \phi\theta\lambda E_\gamma(\omega', w_r)) \\ &\leq \frac{\phi\theta\lambda E_\gamma(\omega', w_r)}{\theta E_\gamma(g\omega', gw_r)} \\ &< \frac{\theta\lambda E_\gamma(\omega', w_r)}{\theta E_\gamma(g\omega', gw_r)}.\end{aligned}$$

Thus, $\gamma(\omega', w_r)\theta E_\gamma(g\omega', gw_r) < \theta\lambda E_\gamma(\omega', w_r)$
and so, $E_\gamma(g\omega', w_{r+1}) = E_\gamma(g\omega', gw_r) < \dfrac{\lambda}{\gamma(\omega', w_r)} E_\gamma(\omega', w_r) < E_\gamma(\omega', w_r)$
Letting $r \to +\infty$, we get $\lim_{r\to+\infty} g\omega' = \omega'$.

Corollary 5: Suppose $(\mathscr{W}, E_\gamma)$ is complete and γ is bounded by $\frac{1}{\lambda}$. Suppose there is $\lambda \in (0,1)$ such that given $g : \mathscr{W} \to \mathscr{W}$ satisfies the following condition:

$$1 \leq e^{\lambda^3 E_\gamma(w_1,w_2) - \ln(\gamma(w_1,w_2))E_\gamma(gw_1,gw_2)} \; for\ all\ w_1, w_2 \in \mathscr{W}. \tag{11}$$

Then there is a unique element in $\mathscr{G}$.

Proof 5: Define $h : [1,+\infty) \times [1,+\infty) \to \mathbb{R}$ via $h(w_1, w_2) = \frac{w_2^\lambda}{w_1}$, $\theta : [0,+\infty) \to [1,+\infty)$ via $\theta(w) = e^w$ and $\phi : [1,+\infty) \to [1,+\infty)$ by $\phi(w) = w^\lambda$. Then, $h \in \mathscr{H}$, $\theta \in \Theta$ and $\phi \in \Phi$. To show that g is $(\gamma, \theta, \phi)_{\mathscr{H}}$-contraction.

From condition 11, we have

$$\begin{aligned}&1 \leq e^{\lambda^3 E_\gamma(w_1,w_2) - \ln(\gamma(w_1,w_2))E_\gamma(gw_1,gw_2)} \\ &\implies 1 \leq \frac{e^{\lambda^3 E_\gamma(w_1,w_2)}}{\gamma(w_1,w_2)e^{E_\gamma(gw_1,gw_2)}} \\ &\implies \gamma(w_1,w_2) \leq \frac{e^{\lambda^3 E_\gamma(w_1,w_2)}}{e^{E_\gamma(gw_1,gw_2)}} \\ &\implies \gamma(w_1,w_2) \leq \frac{(e^{\lambda E_\gamma(w_1,w_2)})^{\lambda^2}}{e^{E_\gamma(gw_1,gw_2)}}\end{aligned}$$

$$\implies \gamma(w_1,w_2) \leq \frac{(\phi\theta\lambda E_\gamma(w_1,w_2))^\lambda}{\theta E_\gamma(gw_1,gw_2)}$$

$$\implies \gamma(w_1,w_2) \leq h\left(\theta E_\gamma(gw_1,gw_2), \phi\theta\lambda E_\gamma(w_1,w_2)\right).$$

Thus, the result comes from theorem 4.

Corollary 6: Suppose $(\mathscr{W}, E_\gamma)$ is complete and γ is bounded by $\frac{1}{\lambda}$. Suppose there is $\lambda \in (0,1)$ such that given $g : \mathscr{W} \to \mathscr{W}$ satisfies the following condition:

$$\ln(\gamma(w_1,w_2))E_\gamma(gw_1,gw_2) \leq \lambda^3 E_\gamma(w_1,w_2) \; for \; all \; w_1,w_2 \in \mathscr{W}. \quad (12)$$

Then there is a unique element in $\mathscr{G}$.

Proof 6: The proof of this corollary comes from corollary 5.

Corollary 7: Suppose $(\mathscr{W}, E_\gamma)$ is complete and γ is bounded by $\frac{1}{\lambda}$. Suppose there are $\lambda \in (0,1)$ and $\epsilon > 0$ such that given $g : \mathscr{W} \to \mathscr{W}$ satisfies the following condition:

$$\epsilon\gamma(w_1,w_2) \leq 2^{\lambda^2 E_\gamma(w_1,w_2)}\left[2^{\lambda^2 E_\gamma(w_1,w_2)} - 2^{E_\gamma(gw_1,gw_2)}\gamma(w_1,w_2)\right] \; for \; all \; w_1,w_2 \in \mathscr{W}. \quad (13)$$

Then there is a unique element in $\mathscr{G}$.

Proof 7: Define $h : [1,+\infty) \times [1,+\infty) \to \mathbb{R}$ via $h(w_1,w_2) = \frac{(w_2)^2}{\epsilon + w_1 w_2}$, $\theta : [0,+\infty) \to [1,+\infty)$ via $\theta(w) = 2^w$ and $\phi : [1,+\infty) \to [1,+\infty)$ by $\phi(w) = w^\lambda$. Then, $h \in \mathscr{H}$, $\theta \in \Theta$ and $\phi \in \Phi$. To show that g is $(\gamma,\theta,\phi)_{\mathscr{H}}$-contraction.

From condition 13,

$$\epsilon\gamma(w_1,w_2) \leq 2^{\lambda^2 E_\gamma(w_1,w_2)}\left[2^{\lambda^2 E_\gamma(w_1,w_2)} - 2^{E_\gamma(gw_1,gw_2)}\gamma(w_1,w_2)\right]$$

$$\implies \epsilon\gamma(w_1,w_2) + 2^{E_\gamma(gw_1,gw_2)+\lambda^2 E_\gamma(w_1,w_2)}\gamma(w_1,w_2) \leq 2^{2\lambda^2 E_\gamma(w_1,w_2)}$$

$$\implies \gamma(w_1,w_2) \leq \frac{2^{2\lambda^2 E_\gamma(w_1,w_2)}}{\epsilon + 2^{E_\gamma(gw_1,gw_2)+\lambda^2 E_\gamma(w_1,w_2)}},$$

$$\implies \gamma(w_1,w_2) \leq \frac{(\phi\theta\lambda E_\gamma(w_1,w_2))^2}{\epsilon + \theta E_\gamma(gw_1,gw_2)\phi\theta\lambda E_\gamma(w_1,w_2)}$$

$$\implies \gamma(w_1,w_2) \leq h\left(\theta E_\gamma(gw_1,gw_2), \phi\theta\lambda E_\gamma(w_1,w_2)\right).$$

Thus, the result comes from theorem 4.

Corollary 8: Suppose $(\mathscr{W}, E_\gamma)$ is complete and γ is bounded by $\frac{1}{\lambda}$. Suppose there is $\lambda \in (0,1)$ such that given $g : \mathscr{W} \to \mathscr{W}$ satisfies the following condition:

$$\gamma(w_1,w_2) - 1 \leq \frac{\lambda}{2}E_\gamma(w_1,w_2) - E_\gamma(gw_1,gw_2) \; for \; all \; w_1,w_2 \in \mathscr{W}. \quad (14)$$

Then there is a unique element in $\mathscr{G}$. From corollary 8, we establish the following corollary:

Corollary 9: Suppose $(\mathscr{W}, E_\gamma)$ is complete. Suppose there is $\tau \in (0, \frac{1}{2})$ such that given $g : \mathscr{W} \to \mathscr{W}$ satisfies the following condition:

$$E_\gamma(gw_1,gw_2) \leq \tau E_\gamma(w_1,w_2) \; for \; all \; w_1,w_2 \in \mathscr{W}. \quad (15)$$

Then there is a unique element in $\mathscr{G}$.

Now, we highlight our results by introducing some examples.
Example 2: Suppose $\mathscr{W} = [0,1]$. Define g on $\mathscr{W}$ via $g(w) = \frac{1}{48}(w^6 + w^4 + w^2 + 1)$. To prove that there is a unique element in $\mathscr{G}$. Let $\gamma : \mathscr{W} \times \mathscr{W} \to [1,2]$ be defined by $\gamma(w_1,w_2) = \frac{1+3w_1w_2}{1+w_1w_2}$ and $E_\gamma : \mathscr{W} \times \mathscr{W} \to [0,+\infty)$ by $E_\gamma(w_1,w_2) = \frac{1}{2}\gamma(w_1,w_2)(w_1-w_2)^2$. Then, it is clear that E_γ is a complete. Moreover, define $h : [1,+\infty) \times [1,+\infty) \to \mathbb{R}$ by $h(w,w') = 1 + \ln\left(\frac{w'}{w}\right)$ and $\theta : [0,+\infty) \to [1,+\infty)$ via $\theta(w^{'}) = e^{w^{'}}$ and $\phi : [1,+\infty) \to [1,+\infty)$ by $\phi(w^{'}) = \sqrt[4]{w'}$. Then $h \in \mathscr{H}$, $\phi \in \Phi$ and $\theta \in \Theta$. To show that there is a unique element in$\mathscr{G}$, want to show that:

$$\gamma(w_1,w_2) \le h\left(\theta E_\gamma(gw_1,gw_2), \phi\theta\lambda E_\gamma(w_1,w_2)\right) \ for\ all\ \ w_1,w_2 \in \mathscr{W}\ and\ \lambda = \frac{1}{2}. \quad (16)$$

Now, for all $w_1, w_2 \in \mathscr{W}$ we have:

$$\begin{aligned} E_\gamma(gw_1,gw_2) &= \tfrac{1}{2}\gamma(gw_1,gw_2)(gw_1-gw_2)^2 \\ &\le \frac{1}{48^2}\left(w_1^6+w_1^4+w_1^2+1-w_2^6-w_2^4-w_2^2-1\right)^2 \\ &\le \frac{1}{16}(w_1-w_2)^2 \\ &= \frac{1}{8\gamma(w_1,w_2)}E_\gamma(w_1,w_2) \\ &= \frac{\lambda}{4\gamma(w_1,w_2)}E_\gamma(w_1,w_2). \end{aligned}$$

Therefore,

$$\gamma(w_1,w_2) \le \frac{\frac{\lambda}{4}E_\gamma(w_1,w_2)}{E_\gamma(gw_1,gw_2)},$$

$$\Longrightarrow \gamma(w_1,w_2) \le 1 + \ln\left(\frac{e^{\frac{\lambda}{4}E_\gamma(w_1,w_2)}}{e^{E_\gamma(gw_1,gw_2)}}\right),$$

$$\Longrightarrow \gamma(w_1,w_2) \le 1 + \ln\left(\frac{\phi\theta\lambda E_\gamma(w_1,w_2)}{\theta E_\gamma(gw_1,gw_2)}\right),$$

$$\Longrightarrow \gamma(w_1,w_2) \le h\left(\theta E_\gamma(gw_1,gw_2), \phi\theta\lambda E_\gamma(w_1,w_2)\right).$$

Consequently, theorem 4 ensures that there is a unique element in $\mathscr{G}$.
Example 3: Consider the following self mapping $g : [0,1] \to [0,1]$ via

$$g(w) = \frac{1+w^5}{5\sqrt{2}+\sqrt{2}w^5}.$$

Then there is a unique element in $\mathscr{G}$.

Proof 8: To prove this, let $\mathscr{W} = [0,1]$. Define $h : [1,+\infty) \times [1,+\infty) \to \mathbb{R}$ via $h(w,w') = \frac{(w')^{\frac{3\sqrt{2}}{5}}}{w}$. Also, define $\gamma : \mathscr{W} \times \mathscr{W} \to [1,\frac{3}{2}]$ by $\gamma(w_1,w_2) = \frac{1+2w_1w_2}{1+w_1w_2}$ and $E_\gamma : \mathscr{W} \times \mathscr{W} \to [0,+\infty)$ by $E_\gamma(w_1,w_2) = \frac{1}{2}\gamma(w_1,w_2)(w_1-w_2)^2$. On the other hand, define $\theta : [0,+\infty) \to [1,+\infty)$ by $\theta(w) = a^w$ where $a > 1$ and $\phi : [1,+\infty) \to [1,+\infty)$ via $\phi(w) = w^{\frac{3\sqrt{2}}{5}}$. It is obviously that, $h \in \mathscr{H}$, E_γ is complete, $\phi \in \Phi$ and $\theta \in \Theta$. To prove that there is a unique element in $\mathscr{G}$, it is suffices to show that

$$\gamma(w_1,w_2) \le \frac{\left(\phi\theta\lambda E_\gamma(w_1,w_2)\right)^{\frac{3\sqrt{2}}{5}}}{\theta E_\gamma(gw_1,gw_2)} \ \ for\ all\ \ w_1,w_2 \in \mathscr{W}\, and\ \lambda = \frac{2}{3}.$$

Now,

$$\begin{aligned} E_\gamma(gw_1, gw_2) &= \frac{1}{2}\gamma(gw_1, gw_2)(gw_1 - gw_2)^2 \\ &\leq \frac{3}{4}\left(\frac{1+w_1^5}{5\sqrt{2}+\sqrt{2}w_1^5} - \frac{1+w_2^5}{5\sqrt{2}+\sqrt{2}w_2^5}\right)^2 \\ &= \frac{3}{8(5+w_1^5)^2(5+w_2^5)^2}\left(4(w_1^5 - w_2^5)\right)^2 \\ &\leq \frac{6}{25}(w_1 - w_2)^2 \\ &= \frac{18\lambda}{25\gamma(w_1, w_2)} E_\gamma(w_1, w_2). \end{aligned}$$

$$\gamma(w_1, w_2) \leq a^{\gamma(w_1,w_2)} \leq \frac{a^{\frac{3\sqrt{2}\lambda}{5}E_\gamma(w_1,w_2)}}{a^{E_\gamma(gw_1,gw_2)}} = \frac{(\phi\theta\lambda E_\gamma(w_1,w_2))^{\frac{3\sqrt{2}}{5}}}{\theta E_\gamma(gw_1, gw_2)}.$$

Consequently, theorem informs us that there is a unique element in $\mathscr{G}$.

3. APPLICATION

In this section, we highlight the the importance of our results by introducing an application. The 17 meets all the expectations of the intermediate value theorem in the unit interval, therefore, the equation has a solution, However, by applying our result we confirm that this solution is unique. Theorem 10 For any $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} x^{2k} = \Lambda x \text{ where } 3n(n+1) \leq \Lambda, \tag{17}$$

has a unique solution in the unit interval $I = [0, 1]$.
Proof 9: Let $\mathscr{W} = I$. Define $\gamma : \mathscr{W} \times \mathscr{W} \to [1, \frac{3}{2}]$ by $\gamma(w_1, w_2) = \frac{1+2w_1w_2}{1+w_1w_2}$ and $E_\gamma : \mathscr{W} \times \mathscr{W} \to [0, +\infty)$ by $E_\gamma(w_1, w_2) = \frac{1}{2}\gamma(w_1, w_2)(w_1 - w_2)^2$. Then, it is clear that E_γ is a complete.
Observe that, our equation has a unique solution in $\mathscr{W}$ iff the mapping $g : \mathscr{W} \to \mathscr{W}$ which defined by

$$g(w) = \frac{1}{\Lambda}\sum_{k=0}^{n} w^{2k}$$

has a unique element in $\mathscr{G}$. To show this, we claim that for all $w_1, w_2 \in \mathscr{W}$, we have (18).

$$E_\gamma(gw_1, gw_2) \leq \frac{\lambda}{4\gamma(w_1, w_2)} E_\gamma(w_1, w_2) \text{ with } \lambda = \frac{2}{3}. \tag{18}$$

Now,

$$E_\gamma(gw_1, gw_2) = \frac{1}{2}\gamma(gw_1, gw_2)\left(\frac{1}{\Lambda}\sum_{k=0}^{n}(w_1^{2k} - w_2^{2k})\right)^2 \leq \frac{3}{4\Lambda^2}\left(\sum_{i=1}^{m}(w_1^{2k} - w_2^{2k})\right)^2$$

$$\leq \frac{3}{4\Lambda^2}(w_1 - w_2)^2\left(2(1+2+\cdots+n)\right)^2 = \frac{3n^2(n+1)^2}{4\Lambda^2}(w_1 - w_2)^2 = \frac{3n^2(n+1)^2}{2\Lambda^2\gamma(w_1, w_2)}E_\gamma(w_1, w_2)$$

$$\leq \frac{\lambda}{4\gamma(w_1, w_2)}E_\gamma(w_1, w_2).$$

To complete our proof, define $h : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$ via $h(w_1, w_2) = \dfrac{\sqrt{w_2}}{w_1}$, $\theta : [0, +\infty) \to [1, +\infty)$ via $\theta(w) = e^w$ and $\phi : [1, +\infty) \to [1, +\infty)$ by $\phi(w) = \sqrt{w}$. Then, $h \in \mathscr{H},\ \theta \in \Theta$

and $\phi \in \Phi$. From (18) we have:

$$\gamma(w_1,w_2) \le \frac{\frac{\lambda}{4}E_\gamma(w_1,w_2)}{E_\gamma(gw_1,gw_2)}, \implies \gamma(w_1,w_2) \le e^{\gamma(w_1,w_2)} \le \frac{e^{\frac{\lambda}{4}E_\gamma(w_1,w_2)}}{e^{E_\gamma(gw_1,gw_2)}},$$

$$\implies \gamma(w_1,w_2) \le \frac{\left(e^{\lambda E_\gamma(w_1,w_2)}\right)^{\frac{1}{4}}}{e^{E_\gamma(gw_1,gw_2)}}, \implies \gamma(w_1,w_2) \le \frac{(\phi\theta\lambda E_\gamma(w_1,w_2))^{\frac{1}{2}}}{\theta E_\gamma(gw_1,gw_2)},$$

$$\implies \gamma(w_1,w_2) \le h\left(\theta E_\gamma(gw_1,gw_2), \phi\theta\lambda E_\gamma(w_1,w_2)\right).$$

Consequently, the function g meets all expectations of theorem 4, thus there is a unique element in $\mathscr{G}$.

4. CONCLUSION

In this study, We proved some new fixed point results based on our new contraction namely, $(\gamma,\phi,\theta)_{\mathscr{H}}$-contraction. This contraction combined a set of concepts such as the concept of extended b-metric spaces and the concept of $\mathscr{H}$-simulation functions. Moreover, we showed the applicability of our new results by introducing some numerical examples and we showed the novelty of our results by introducing an application.

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