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ϵ_φ-contraction and some fixed point results via modified ω-distance mappings in the frame of complete quasi metric spaces and applications

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ABSTRACT

In this Article, we introduce the notion of an ϵ_φ-contraction, which is based on modified ω-distance mappings, and employ this new definition to prove some fixed point result. Moreover, to highlight the significance of our work, we present an interesting example along with an application.

1. INTRODUCTION

One of the most important methods in mathematics used to discuss the existence and uniqueness of a solution of such equations is the Banach contraction principle [1]. It is considered as a valuable tool in fixed point theory. Since then, many mathematicians investigated the Banach contraction principle in many directions. In [2], Abodayeh et al. utilized the concept of $\Omega-$distance to give some new generalizations of Banach contraction principle. Shatanawi, M. Postolache in [3, 4] studied some common fixed points of such mappings. For more generalizations of Banach fixed point theory, see [5–18]. In 1931 Wilson [19] introduced the notion of quasi metric space as below:

Definition 1 *[19] We call the function $q : E \times E \to [0, \infty)$ a quasi metric if it satisfies:*

(i) $q(e_1, e_2) = 0 \Leftrightarrow e_1 = e_2$*;*

(ii) $q(e_1, e_2) \leq (e_1, e_3) + q(e_3, e_1)$ *for all* $e_1, e_2, e_3 \in E$*.*

The pair (E, q) is called a quasi metric space.

For some work in quasi metric spaces, see [20–23]

If the symmetry condition is added to (E, q) (i.e. $q(e_1, e_2) = q(e_2, e_1)$ for all $e_1, e_2 \in E$), then the space (E, q) is a metric space.

Henceforth, we denote by (E, q) a quasi metric space.
To generate a metric d on E. Define $d : E \times E \to [0, \infty)$ by

$$d = \max\{q(e_1, e_2), q(e_2, e_1)\}.$$

The concepts of completeness and convergence of quasi metric spaces are given below:

Definition 2 *[24, 25] A sequence (e_s) converges to $e^* \in E$ if $\lim_{s\to\infty} q(e_s, e^*) = \lim_{s\to\infty} q(e^*, e_s) = 0$.*

Definition 3 *[25] Let (e_s) be a sequence in E. Then we call*

(i) (e_s) left-Cauchy if for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that $q(e_s, e_t) < \delta$ for all $s \geq t > N_0$.

(ii) (e_s) right-Cauchy if for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that $q(e_s, e_t) < \delta$ for all $t \geq s > N_0$.

Definition 4 *[24, 25] A sequence (e_s) in E is called a Cauchy sequence if*

*(i) If for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that $q(e_s, e_t) \leq \delta$ for all $s, t > N_0$;
or*

(ii) (e_s) is right and left Cauchy.

Definition 5 *[24, 25] We say (E, q) is complete if every Cauchy sequence (e_s) in E is convergent.*

In 2016, Alegre and Marin [26] introduced the notion of modified ω-distance mappings on (E, q).

Definition 6 *[26] A modified ω-distance (shortly mω-distance) on (E, q) is a function $\rho : E\times E \to [0, \infty)$, which satisfies the following:*

(W1) $\rho(e_1, e_2) \leq \rho(e_1, e_3) + \rho(e_3, e_2)$ for all $e_1, e_2, e_3 \in E$;

(W2) $\rho(e, .) : E \to [0, \infty)$ is lower semi-continuous for all $e \in E$; and

(mW3) for each $\varrho > 0$ there exists $\delta > 0$ such that if $\rho(e_1, e_2) \leq \delta$ and $\rho(e_2, e_3) \leq \delta$, then $q(e_1, e_3) \leq \varrho$ for all $e_1, e_2, e_3 \in E$.

Henceforth, we denote by ρ an mω-distance mapping.

Definition 7 *[26] if ρ is lower semi-continuous on the first and second coordinates, then ρ is called a strong mω-distance.*

Remark 1 *[26] Every quasi metric q on E is mω-distance.*

Lemma 1 *[33] Let (ϱ_s), (σ_s) be two sequences of nonnegative real numbers that converge to zero. Then we have the following:*

(i) If $\rho(e_s, e_t) \leq \varrho_s$ for all $s, t \in \mathbb{N}$ with $t \geq s$, then (e_s) is right Cauchy in (E, q).

(ii) If $\rho(e_s, e_t) \leq \sigma_t$ for all $s, t \in \mathbb{N}$ with $t \leq s$, then (e_s) is left Cauchy in (E, q).

Remark 2 *[33] The above lemma show that if $\lim_{s,t\to\infty} p(e_s, e_t) = 0$, then (e_s) is Cauchy in (E, q).*

For more results in fixed point theory in ω and modified $\omega-$distances, we ask the readers to consider [20, 27–31, 33, 34].

Definition 8 *[35] A self function φ on $[0, \infty)$ is said to be an ultra distance function if φ satisfies $\varphi(\mu^*) = 0 \Leftrightarrow \mu^* = 0$ and if (μ^*_s) is a sequence in $[0, \infty)$ such that $\lim_{s\to+\infty} \varphi(\mu^*_s) = 0$, then $\lim_{s\to+\infty} \mu^*_s = 0$.*

2. MAIN RESULTS

The definition of ϵ_φ-contraction on a pair of self mappings is defined as follows:

Definition 9 *Equipped (E, q) with ρ and let F, T be two self mappings on E. Then the pair (F, T) is called ϵ_φ-contraction if there exists an ultra distance function φ and a given $\epsilon > 0$ such that for all $e_1, e_2 \in E$ we have:*

$$\varphi\rho(Fe_1, Te_2) \leq \left(\frac{\rho(e_1, e_2)}{\epsilon + \rho(e_1, Fe_1)}\right) \max\Big\{\varphi\rho(e_1, Fe_1), \varphi\rho(e_2, Te_2)\Big\}.$$

And

$$\varphi\rho(Te_1, Fe_2) \leq \left(\frac{\rho(e_1, e_2)}{\epsilon + \rho(e_1, Te_1)}\right) \max\Big\{\varphi\rho(e_1, Te_1), \varphi\rho(e_2, Fe_2)\Big\}.$$

Next, we introduce our first result:

Theorem 2 *Equipped (E, q) with ρ and let F, T be two self mappings on E such that the pair (F, T) is an ϵ_φ-contraction. Also, assume $\rho(e_{j+1}, e_j) = 0$ or $\rho(e_j, e_{j+1}) = 0$, for some $j \in \mathbb{N} \cup \{0\}$. Then e_j is a unique common fixed point of F and T in E.*

Proof. Let $e_0 \in E$. We create a sequence (e_j) in E inductively by taking $Fe_{2j} = e_{2j+1}$ and $Te_{2j+1} = e_{2j+2}$ for all $j \in \mathbb{N} \cup \{0\}$.
To prove the result, we have to consider the following cases:
Case(1): $\rho(e_j, e_{j+1}) = 0$. If j is even, then $j = 2k$ for some $k \in \mathbb{N} \cup \{0\}$, so we have $\rho(e_{2k}, e_{2k+1}) = 0$ and so $\varphi\rho(e_{2k}, e_{2k+1}) = 0$.
Now, since the pair (F, T) is an ϵ_φ-contraction, we get:

$$\begin{aligned}
\varphi\rho(e_{2k+1}, e_{2k+2}) &= \varphi\rho(Fe_{2k}, Te_{2k+1}) \\
&\leq \left(\tfrac{\rho(e_{2k}, e_{2k+1})}{\epsilon + \rho(e_{2k}, Fe_{2k})}\right) \max\Big\{\varphi\rho(e_{2k}, Fe_{2k}), \varphi\rho(e_{2k+1}, Te_{2k+1})\Big\} \\
&= \left(\tfrac{\rho(e_{2k}, e_{2k+1})}{\epsilon + \rho(e_{2k}, e_{2k+1})}\right) \max\Big\{\varphi\rho(e_{2k}, e_{2k+1}), \varphi\rho(e_{2k+1}, e_{2k+2})\Big\} \\
&= 0.
\end{aligned}$$

By the definition of φ, we have

$$\rho(e_{2k+1}, e_{2k+2}) = 0. \tag{1}$$

From the assumption we have $\rho(e_{2k}, e_{2k+1}) = 0$ and by (1) we get that

$$\rho(e_{2k}, e_{2k+2}) = 0. \tag{2}$$

Also, by using mW3 of the definition of ρ, we get that

$$q(e_{2k}, e_{2k+2}) = 0. \tag{3}$$

$$\begin{aligned}
\varphi\rho(e_{2k+2}, e_{2k+1}) &= \varphi\rho(Te_{2k+1}, Fe_{2k}) \\
&\leq \left(\tfrac{\rho(e_{2k+1}, e_{2k})}{\epsilon + \rho(e_{2k+1}, Te_{2k+1})}\right) \max\Big\{\varphi\rho(e_{2k+1}, e_{2k+2}), \varphi\rho(e_{2k}, e_{2k+1})\Big\} \\
&= \left(\tfrac{\rho(e_{2k+1}, e_{2k})}{\epsilon + \rho(e_{2k+1}, e_{2k+2})}\right) \max\Big\{\varphi\rho(e_{2k+1}, e_{2k+2}), \varphi\rho(e_{2k}, e_{2k+1})\Big\} \\
&= 0.
\end{aligned}$$

Therefore,

$$\rho(e_{2k+2}, e_{2k+1}) = 0. \tag{4}$$

Also, using the Equations (2), (4) and mW3 of the definition of ρ, we get that

$$q(e_{2k+1}, e_{2k}) = 0. \tag{5}$$

Hence, $e_{2k} = e_{2k+1} = e_{2k+2}$ and so e_j is a common fixed point of F and T in E.

If j is odd, then $j = 2k+1$, for some $k \in \mathbb{N} \cup \{0\}$. Then we have $\rho(e_{2k+1}, e_{2k+2}) = 0$ and hence $\varphi\rho(e_{2k+1}, e_{2k+2}) = 0$.

$$\begin{aligned}\varphi\rho(e_{2k+2}, e_{2k+3}) &= \varphi\rho(Te_{2k+1}, Fe_{2k+2}) \\ &\leq \left(\frac{\rho(e_{2k+1}, e_{2k+2})}{\epsilon + \rho(e_{2k+1}, Te_{2k+1})}\right) \max\left\{\varphi\rho(e_{2k+1}, e_{2k+2}), \varphi\rho(e_{2k+2}, e_{2k+3})\right\} \\ &= \left(\frac{\rho(e_{2k+1}, e_{2k+2})}{\epsilon + \rho(e_{2k+1}, e_{2k+2})}\right) \max\left\{\varphi\rho(e_{2k+1}, e_{2k+2}), \varphi\rho(e_{2k+2}, e_{2k+3})\right\} \\ &= \left(\frac{\rho(e_{2k+1}, e_{2k+2})}{\epsilon + \rho(e_{2k+1}, e_{2k+2})}\right) \varphi\rho(e_{2k+2}, e_{2k+3}).\end{aligned}$$

Let $L = \frac{\rho(e_{2k+1}, e_{2k+2})}{\epsilon + \rho(e_{2k+1}, e_{2k+2})}$. Then $L < 1$ and so

$$\varphi\rho(e_{2k+2}, e_{2k+3}) < \varphi\rho(e_{2k+2}, e_{2k+3}).$$

Thus, $\varphi\rho(e_{2k+2}, e_{2k+3}) = 0$. By the definition φ, we get that

$$\rho(e_{2k+2}, e_{2k+3}) = 0. \tag{6}$$

From the assumption, we have $\rho(e_{2k+1}, e_{2k+2}) = 0$ and by (6), we get

$$\rho(e_{2k+1}, e_{2k+3}) = 0. \tag{7}$$

Also, Condition mW3 of the definition of ρ implies that

$$q(e_{2k+1}, e_{2k+3}) = 0. \tag{8}$$

$$\begin{aligned}\varphi\rho(e_{2k+3}, e_{2k+2}) &= \varphi\rho(Fe_{2k+2}, Te_{2k+1}) \\ &\leq \left(\frac{\rho(e_{2k+2}, e_{2k+1})}{\epsilon + \rho(e_{2k+2}, Fe_{2k+2})}\right) \max\left\{\varphi\rho(e_{2k+2}, Fe_{2k+2}), \varphi\rho(e_{2k+1}, Te_{2k+1})\right\} \\ &= \left(\frac{\rho(e_{2k+2}, e_{2k+1})}{\epsilon + \rho(e_{2k+2}, e_{2k+3})}\right) \max\left\{\varphi\rho(e_{2k+2}, e_{2k+3}), \varphi\rho(e_{2k+1}, e_{2k+2})\right\} \\ &= 0.\end{aligned}$$

In a similar manner, we can prove that if $\rho(e_{j+1}, e_j) = 0$, then e_j is a common fixed point of F and T in E. □

Next, we introduce our main result:

Theorem 3 *Equipped (E, q) with ρ and let F, T be two self mappings on E. Assume the following conditions hold:*
(i) (E, q) is complete;
(ii) The pair (F, T) is an ϵ_φ-contraction ;
(iii) F and T are continuous;
(iv) For all $e_1, e_2 \in E$ and some integer L we have $\rho(e_1, e_2) \leq L$.
Then F and T have a unique common fixed point in E.

Proof. Let $e_0 \in E$. Construct a sequence (e_n) in E inductively by taking $Fe_{2n} = e_{2n+1}$ and $Te_{2n+1} = e_{2n+2}$ for all $n \in \mathbb{N} \cup \{0\}$.
If for some $i \in \mathbb{N}$ we have $\rho(e_i, e_{i+1}) = 0$ or $\rho(e_{i+1}, e_i) = 0$, then by Theorem 2, e_i is a unique common fixed point of F and T in E.

Now, assume that $\rho(e_n, e_{n+1}) \neq 0$ and $\rho(e_{n+1}, e_n) \neq 0$, for all $n \in \mathbb{N} \cup \{0\}$.
Since the pair (F, T) is an ϵ_φ-contraction, then we have

$$\begin{aligned}
\varphi\rho(e_{2n+1}, e_{2n+2}) &= \varphi\rho(Fe_{2n}, Te_{2n+1}) \\
&\leq \left(\frac{\rho(e_{2n}, e_{2n+1})}{\epsilon+\rho(e_{2n}, Fe_{2n})}\right) \max\left\{\varphi\rho(e_{2n}, Fe_{2n}), \varphi\rho(e_{2n+1}, Te_{2n+1})\right\} \\
&= \left(\frac{\rho(e_{2n}, e_{2n+1})}{\epsilon+\rho(e_{2n}, e_{2n+1})}\right) \max\left\{\varphi\rho(e_{2n}, e_{2n+1}), \varphi\rho(e_{2n+1}, e_{2n+2})\right\}.
\end{aligned}$$

If $L = \frac{\rho(e_{2n}, e_{2n+1})}{\epsilon+\rho(e_{2n}, e_{2n+1})}$, then $L < 1$.
Also, if $\max\left\{\varphi\rho(e_{2n}, e_{2n+1}), \varphi\rho(e_{2n+1}, e_{2n+2})\right\} = \varphi\rho(e_{2n+1}, e_{2n+2})$, we get that

$$\begin{aligned}
\varphi\rho(e_{2n+1}, e_{2n+2}) &\leq L \max\left\{\varphi\rho(e_{2n}, e_{2n+1}), \varphi\rho(e_{2n+1}, e_{2n+2})\right\} \\
&= L\varphi\rho(e_{2n+1}, e_{2n+2}) \\
&< \varphi\rho(e_{2n+1}, e_{2n+2}).
\end{aligned} \tag{9}$$

Thus, $\varphi\rho(e_{2n+1}, e_{2n+2}) = 0$ and so $\rho(e_{2n+1}, e_{2n+2}) = 0$ a contradiction.
Therefore,

$$\varphi\rho(e_{2n+1}, e_{2n+2}) \leq (\frac{\rho(e_{2n}, e_{2n+1})}{\epsilon + \rho(e_{2n}, e_{2n+1})})\varphi\rho(e_{2n}, e_{2n+1}). \tag{10}$$

$$\begin{aligned}
\varphi\rho(e_{2n+2}, e_{2n+1}) &= \varphi\rho(Te_{2n+1}, Fe_{2n}) \\
&\leq \left(\frac{\rho(e_{2n+1}, e_{2n})}{\epsilon+\rho(e_{2n+1}, Te_{2n+1})}\right) \max\left\{\varphi\rho(e_{2n+1}, e_{2n+2}), \varphi\rho(e_{2n}, e_{2n+1})\right\} \\
&= \left(\frac{\rho(e_{2n+1}, e_{2n})}{\epsilon+\rho(e_{2n+1}, e_{2n+2})}\right) \max\left\{\varphi\rho(e_{2n+1}, e_{2n+2}), \varphi\rho(e_{2n}, e_{2n+1})\right\} \\
&= \left(\frac{\rho(e_{2n+1}, e_{2n})}{\epsilon+\rho(e_{2n+1}, e_{2n+2})}\right)\varphi\rho(e_{2n}, e_{2n+1}).
\end{aligned}$$

Also, we can show that:

$$\varphi\rho(e_n, e_{n+1}) \leq (\frac{\rho(e_{n-1}, e_n)}{\epsilon + \rho(e_{n-1}, e_n)})\varphi\rho(e_{n-1}, e_n). \tag{11}$$

And

$$\varphi\rho(e_{n+1}, e_n) \leq \left(\frac{\rho(e_n, e_{n-1})}{\epsilon + \rho(e_n, e_{n+1})}\right)\varphi\rho(e_{n-1}, e_n). \tag{12}$$

Now,

$$\begin{aligned}
\varphi\rho(e_n, e_{n+1}) &\leq \left(\frac{\rho(e_{n-1}, e_n)}{\epsilon+\rho(e_{n-1}, e_n)}\right)\varphi\rho(e_{n-1}, e_n) \\
&\leq \left(\frac{\rho(e_{n-1}, e_n)}{\epsilon+\rho(e_{n-1}, e_n)}\right)\left(\frac{\rho(e_{n-2}, e_{n-1})}{\epsilon+\rho(e_{n-2}, e_{n-1})}\right)\varphi\rho(e_{n-2}, e_{n-1}) \\
&\leq \cdots \leq \prod_{i=1}^{n} \left(\frac{\rho(e_{i-1}, e_i)}{\epsilon+\rho(e_{i-1}, e_i)}\right)\varphi\rho(e_0, e_1).
\end{aligned}$$

let $L_i = (\frac{\rho(e_{i-1}, e_i)}{\epsilon+\rho(e_{i-1}, e_i)})$. Then $L_i < 1$ for all $i \in \{1, 2, \cdots, n\}$, so we have

$$\varphi\rho(e_n, e_{n+1}) \leq \prod_{i=1}^{n-1} L_i(\varphi\rho(e_n, e_{n+1})). \tag{13}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \varphi\rho(e_n, e_{n+1}) = 0. \tag{14}$$

Since φ is ultra distance function, we have

$$\lim_{n\to\infty} \rho(e_n, e_{n+1}) = 0. \tag{15}$$

$$\begin{aligned}
\varphi\rho(e_{n+1}, e_n) &\leq \left(\frac{\rho(e_n,e_{n-1})}{\epsilon+\rho(e_n,e_{n+1})}\right)\varphi\rho(e_{n-1}, e_n)\\
&\leq \left(\frac{\rho(e_n,e_{n-1})}{\epsilon+\rho(e_n,e_{n+1})}\right)\left(\frac{\rho(e_{n-2},e_{n-1})}{\epsilon+\rho(e_{n-2},e_{n-1})}\right)\varphi\rho(e_{n-2}, e_{n-1})\\
&\leq \cdots \leq \left(\frac{\rho(e_n,e_{n-1})}{\epsilon+\rho(e_n,e_{n+1})}\right)\prod_{i=1}^{n-1}\left(\frac{\rho(e_{i-1},e_i)}{\epsilon+\rho(e_i,e_{i-1})}\right)\varphi\rho(e_0, e_1).
\end{aligned}$$

Let $L_i = (\frac{\rho(e_{i-1},e_i)}{\epsilon+\rho(e_{i-1},e_i)})$. Then $L_i < 1$ for all $i \in \{1, 2, \cdots, n-1\}$ and since $\rho(e_1, e_2) \leq L$ for all $e_1, e_1 \in E$ and some integer L, we get that

$$\varphi\rho(e_{n+1}, e_n) \leq L \prod_{i=1}^{n-1} L_i(\varphi\rho(e_0, e_1)). \tag{16}$$

Letting $n \to \infty$, we get that:

$$\lim_{n\to\infty} \varphi\rho(e_{n+1}, e_n) = 0. \tag{17}$$

The definition of φ informs us

$$\lim_{n\to\infty} \rho(e_{n+1}, e_n) = 0. \tag{18}$$

Now, we need to show that (e_s) is a Cauchy sequence in E.
In order to do that, we first prove that (e_s) is a right Cauchy sequence in (E, q). For each $s, t \in \mathbb{N}$ with $s < t$, we have the following cases:
Case (1): If s odd and t even, then we have:

$$\begin{aligned}
\varphi\rho(e_s, e_t) &= \varphi\rho(Fe_{s-1}, Te_{t-1})\\
&\leq \left(\frac{\rho(e_{s-1},e_{t-1})}{\epsilon+\rho(e_{s-1},Fe_{s-1})}\right)\max\left\{\varphi\rho(e_{s-1}, Fe_{s-1}), \varphi\rho(e_{t-1}, Te_{t-1})\right\}\\
&= \left(\frac{\rho(e_{s-1},e_{t-1})}{\epsilon+\rho(e_{s-1},e_s)}\right)\max\left\{\varphi\rho(e_{s-1}, e_s), \varphi\rho(e_{t-1}, e_t)\right\}.\\
&= \left(\frac{\rho(e_{s-1},e_{t-1})}{\epsilon+\rho(e_{s-1},e_s)}\right)\varphi\rho(e_{s-1}, e_s).
\end{aligned}$$

Let $L_i = (\frac{\rho(e_{i-1},e_i)}{\epsilon+\rho(e_{i-1},e_i)})$. Since $\rho(e_1, e_2) \leq L$ for all $e_1, e_2 \in E$ and some integer L, we have

$$\varphi\rho(e_s, e_t) \leq L \prod_{i=1}^{s-1} L_i(\varphi\rho(e_0, e_1)). \tag{19}$$

Letting $s, t \to \infty$, we have $\lim\limits_{s,t\to\infty} \varphi\,(\rho(e_s, e_t)) = 0$.
Thus,

$$\lim_{s,t\to\infty} \varphi\,\rho(e_s, e_t) = 0. \tag{20}$$

Case (2): If s even and t odd, then we have:

$$\begin{aligned}\varphi\rho(e_s,e_t) &= \varphi\rho(Te_{s-1},Fe_{t-1})\\ &\leq \left(\frac{\rho(e_{s-1},e_{t-1})}{\epsilon+\rho(e_{s-1},Te_{s-1})}\right)\max\left\{\varphi\rho(e_{s-1},Te_{s-1}),\varphi\rho(e_{t-1},Fe_{t-1})\right\}\\ &= \left(\frac{\rho(e_{s-1},e_{t-1})}{\epsilon+\rho(e_{s-1},e_s)}\right)\max\left\{\varphi\rho(e_{s-1},e_s),\varphi\rho(e_{t-1},e_t)\right\}.\\ &= \left(\frac{\rho(e_{s-1},e_{t-1})}{\epsilon+\rho(e_{s-1},e_s)}\right)\varphi\rho(e_{s-1},e_s).\end{aligned}$$

Let $L_i = (\frac{\rho(e_{i-1},e_i)}{\epsilon+\rho(e_{i-1},e_i)})$. Since $\rho(e_1,e_2)\leq L$ for all $e_1,e_2\in E$ and some integer L, then we get that

$$\varphi\rho(e_s,e_t)\leq L\prod_{i=1}^{s-1}L_i(\varphi\rho(e_0,e_1)). \tag{21}$$

Letting $s,t\to\infty$, we have $\lim\limits_{s,t\to\infty}\varphi\left(\rho(e_s,e_t)\right)=0$.

So,

$$\lim_{s,t\to\infty}\varphi\,\rho(e_s,e_t)=0. \tag{22}$$

Case (3): If s and t are odd, we get

$$\rho(e_s,e_t)\leq\rho(e_s,e_{s+1})+\rho(e_{s+1},e_t). \tag{23}$$

Hence,

$$\lim_{s,t\to\infty}\rho(e_s,e_t)=0. \tag{24}$$

Case (4): If s and t are even, we get

$$\rho(e_s,e_t)\leq\rho(e_s,e_{t-1})+\rho(e_{t-1},e_t). \tag{25}$$

Hence,

$$\lim_{s,t\to\infty}\rho(e_s,e_t)=0. \tag{26}$$

Using Lemma 1, we get that (e_s) is a right Cauchy sequence in (E,q). Similarly, we can prove that (e_s) is a left Cauchy sequence in E.
Hence, (e_s) is a Cauchy sequence in E. The completeness of (E,q) implies that there exists an element $e^*\in E$ such that $(e_s)\to e^*$.
If F is a continuous function then $e_{s+1}=Fe_s\to Fe^*$. By the uniqueness of limit, we get that $Fe^*=e^*$.
In a similar manner, we can prove that $Te^*=e^*$ when T is a continuous function.
To prove the uniqueness of e^*. First we show that $\rho(e^*,e^*)=0$.

$$\begin{aligned}\varphi\rho(e^*,e^*) &= \varphi\rho(Fe^*,Te^*)\\ &\leq \left(\frac{\rho(e^*,e^*)}{\epsilon+\rho(e^*,Fe^*}\right)\max\left\{\varphi\rho(e^*,Fe^*),\varphi\rho(e^*,Te^*)\right\}\\ &= \left(\frac{\rho(e^*,e^*)}{\epsilon+\rho(e^*,e^*)}\right)\max\left\{\varphi\rho(e^*,e^*),\varphi\rho(e^*,e^*)\right\}\\ &= 0.\end{aligned}$$

Therefore, $\rho(e^*,e^*)=0$.

Assume that there exists $\mu^* \in E$ such that $F\mu^* = T\mu^* = \mu^*$. Then

$$\begin{aligned}\varphi\rho(e^*, \mu^*) &= \varphi\rho(Fe^*, T\mu^*)\\ &\leq \left(\frac{\rho(e^*,\mu^*)}{\epsilon+\rho(e^*,Fe^*}\right)\max\left\{\varphi\rho(e^*, Fe^*), \varphi\rho(\mu^*, T\mu^*)\right\}\\ &= \left(\frac{\rho(e^*,\mu^*)}{\epsilon+\rho(e^*,e^*)}\right)\max\left\{\varphi\rho(e^*, e^*), \varphi\rho(\mu^*, \mu^*)\right\}\\ &= 0.\end{aligned}$$

Thus, we have $\rho(e^*, \mu^*) = 0$ since $\rho(e^*, e^*) = 0$ we get that $q(e^*, \mu^*) = 0$ and so $e^* = \mu^*$. □

Corollary 4 *A complete (E, q) Equipped with ρ and let F, T be two self continuous mappings on E. Assume the following conditions hold:*
(i) For all $e_1, e_2 \in E$ and a given $\epsilon > 0$ and an ultra distance function φ we have:

$$\varphi\rho(Fe_1, Te_2) \leq \left(\frac{\rho(e_1, e_2)}{2(\epsilon + \rho(e_1, Fe_1))}\right)\Big(\varphi\rho(e_1, Fe_1) + \varphi\rho(e_2, Te_2)\Big).$$

And

$$\varphi\rho(Te_1, Fe_2) \leq \left(\frac{\rho(e_1, e_2)}{2(\epsilon + \rho(e_1, Te_1))}\right)\Big(\varphi\rho(e_1, Te_1) + \varphi\rho(e_2, Fe_2)\Big).$$

(ii) For all $e_1, e_2 \in E$ we have $\rho(e_1, e_2) \leq L$ for some integer L.

Then F and T have a unique common fixed point in E.

Proof.

$$\begin{aligned}\varphi\rho(Fe_1, Te_2) &\leq \left(\frac{\rho(e_1,e_2)}{2(\epsilon+\rho(e_1,Fe_1))}\right)\Big(\varphi\rho(e_1, Fe_1) + \varphi\rho(e_2, Te_2)\Big)\\ &\leq \left(\frac{\rho(e_1,e_2)}{\epsilon+\rho(e_1,Fe_1)}\right)\max\left\{\varphi\rho(e_1, Fe_1), \varphi\rho(e_2, Te_2)\right\}.\end{aligned}$$

Similarly, we can prove that:

$$\varphi\rho(Te_1, Fe_2) \leq \left(\frac{\rho(e_1, e_2)}{2(\epsilon + \rho(e_1, Te_1)}\right)\Big(\varphi\rho(e_1, Te_1) + \varphi\rho(e_2, Fe_2)\Big).$$

□

Corollary 5 *A complete (E, q) Equipped with ρ and let F, T be two self continuous mappings on E. Assume the following conditions hold:*
(i) For all $e_1, e_2 \in E$ and for a given $\epsilon > 0$ and an ultra distance function φ and $k \in [0, 1)$ we have:

$$\varphi\rho(Fe_1, Te_2) \leq k\varphi\rho(e_1, e_2).$$

And

$$\varphi\rho(Te_1, Fe_2) \leq k\varphi\rho(e_1, e_2).$$

(ii) For all $e_1, e_2 \in E$ we have $\rho(e_1, e_2) \leq L$ for some integer L.

Then F and T have a unique common fixed point in E.

Proof. Let $\varphi(\mu_*) = \mu_*$ and let $k = (\frac{\rho(e_1,Fe_1)}{\epsilon+\rho(e_1,Fe_1)})$. Then $k \in [0,1)$.
Now,

$$\begin{aligned}
\varphi\rho(Fe_1,Te_2) &= \rho(Fe_1,Te_2)\\
&\leq \left(\frac{\rho(e_1,Fe_1)}{\epsilon+\rho(e_1,Fe_1)}\right)\rho(e_1,e_2)\\
&= \left(\frac{\rho(e_1,e_2)}{\epsilon+\rho(e_1,Fe_1)}\right)\rho(e_1,Fe_1)\\
&= \left(\frac{\rho(e_1,e_2)}{\epsilon+\rho(e_1,Fe_1)}\right)\varphi\rho(e_1,Fe_1)\\
&\leq \left(\frac{\rho(e_1,e_2)}{\epsilon+\rho(e_1,Fe_1)}\right)\max\left\{\varphi\rho(e_1,Fe_1),\varphi\rho(e_2,Te_2)\right\}.
\end{aligned}$$

Similarly, we can prove that:

$$\varphi\rho(Te_1,Fe_2) \leq k\varphi\rho(e_1,e_2).$$

□

If we take $F = T$ in Corollary 5, we get the following result:

Corollary 6 *A complete (E,q) Equipped with ρ and let F be a self continuous mapping on E. Assume the following conditions hold:*
(i) For all $e_1,e_2 \in E$ and for a given $\epsilon > 0$ and an ultra distance function φ and $k \in [0,1)$we have:

$$\varphi\rho(Fe_1,Fe_2) \leq k\varphi\rho(e_1,e_2).$$

(ii) For all $e_1,e_2 \in E$ we have $\rho(e_1,e_2) \leq L$ for some integer L.

Then F has a unique common fixed point in E.

Example 1 *Let $E = 0,1,\cdots,m$ where $m \in \mathbb{N}$.*
Define F,T on E as follows:

$$F(e_1) = \begin{cases} 0 & \textit{if } e_1 \in \{0,1\};\\ 1 & \textit{if } e_1 \in \{2,3,\cdots,5\};\\ 2 & \textit{if } e_1 \in \{6,7,\cdots,m\}.\end{cases}$$

$$T(e_2) = \begin{cases} 0 & \textit{if } e_2 \in \{0,1,\cdots,5\};\\ 1 & \textit{if } e_2 \in \{6,7,\cdots,10\};\\ 2 & \textit{if } e_2 \in \{11,12,\cdots,m\}.\end{cases}$$

Then F and T have a unique fixed point in E.

Proof. *To show that F and T have a unique fixed point in E.*
Define $\rho,q : E\times E \to [0,\infty)$ such that

$$q(e_1,e_2) = \frac{2}{3}e_1 + \frac{1}{3}e_2.$$

$$\rho(e_1,e_2) = 2e_1 + e_2.$$

Also define $\varphi(\mu_*) : [0,\infty) \to [0,\infty)$ as follows:

$$\varphi(\mu_*) = \begin{cases} (1/4)\mu_* & \textit{if } \mu_* \in [0,m];\\ (1/4)(\mu_*^2+2) & \textit{if } \mu_* > m.\end{cases}$$

Then

1. *F and T are continuous functions.*

2. *φ is an ultra distance function.*

3. *(E, q) is a complete quasi metric space.*

4. *ρ is an $m\omega$-distance mapping.*

5. *The pair (F, T) is ϵ_φ-contraction with ($\epsilon = 1$)*
   i.e., $\forall e_1, e_2 \in E$ we have

$$\varphi\rho(Fe_1, Te_2) \leq (\frac{\rho(e_1, e_2)}{1+\rho(e_1, Fe_1)}) \max\Big\{\varphi\rho(e_1, Fe_1), \varphi\rho(e_2, Te_2)\Big\}.$$

And

$$\varphi\rho(Te_1, Fe_2) \leq (\frac{\rho(e_1, e_2)}{1+\rho(e_1, Te_1)}) \max\Big\{\varphi\rho(e_1, Te_1), \varphi\rho(e_2, Fe_2)\Big\}.$$

Now, it is an easy matter to check out that F and T are continuous functions. In addition,, it is obviously that φ is an ultra distance function, ρ is an $m\omega$-distance mapping and (E, q) is a quasi metric space. To show that q is complete, let (e_s) be a Cauchy sequence in E. Then for each $s, t \in \mathbb{N}$ we have

$$\lim_{s,t\to\infty} q(e_s, e_t) = 0$$

we conclude that $e_s = e_t$ for all $s, t \in \mathbb{N}$ but not for finitely many. Therefore, (e_s) is a convergent sequence in E. Consequently, (E, q) is a complete quasi metric space.
To prove that the pair (F, T) is ϵ_φ-contraction with ($\epsilon = 1$), we need to consider the following cases:
Case (1): If $e_1 \in \{0, 1\}$, then we have the following subcases:
Subcase (1): If $e_2 \in \{0, 1, \cdots, 5\}$, then

$$\varphi\rho(Fe_1, Te_2) = \varphi\rho(0, 0) = 0.$$

Subcase (2): If $e_2 \in \{6, 7, \cdots, 10\}$, then

$$\varphi\rho(Fe_1, Te_2) = \varphi\rho(0, 1) = \varphi(1) = \frac{1}{4}.$$

$$\begin{aligned}\left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,Fe_1)}\right)\max\Big\{\varphi\rho(e_1, Fe_1), \varphi\rho(e_2, Te_2)\Big\} &= \left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,0)}\right)\left[\tfrac{1}{4}\rho(e_2, 1)\right]\\ &= \left(\tfrac{2e_1+e_2}{2e_1+1}\right)\left[\tfrac{1}{4}(2e_2+1)\right]\\ &\geq \tfrac{13}{4}\left(\tfrac{2e_1+6}{2e_1+1}\right)\\ &\geq (\tfrac{8}{3})(\tfrac{13}{4})\\ &\geq \tfrac{1}{4}.\end{aligned}$$

Subcase (3): If $e_2 \in \{11, 12, \cdots, m\}$, then we get that

$$\varphi\rho(Fe_1, Te_2) = \varphi\rho(0, 2) = \varphi(2) = \frac{2}{4}.$$

$$\begin{aligned}\left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,Fe_1)}\right)\max\Big\{\varphi\rho(e_1, Fe_1), \varphi\rho(e_2, Te_2)\Big\} &= \left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,0)}\right)\left[\tfrac{1}{4}\rho(e_2, 2)\right]\\ &= \left(\tfrac{2e_1+e_2}{2e_1+1}\right)\left[\tfrac{1}{4}(2e_2+2)\right]\\ &\geq 6\left(\tfrac{2e_1+11}{2e_1+1}\right)\\ &\geq 26\\ &\geq \tfrac{2}{4}.\end{aligned}$$

Case (2): If $e_1 \in \{2, 3, \cdots, 5\}$, then we have the following subcases:
Subcase (1): If $e_2 \in \{0, 1, \cdots, 5\}$, then we have

$$\varphi\rho(Fe_1, Te_2) = \varphi\rho(1, 0) = \varphi(2) = \frac{2}{4}.$$

$$\begin{aligned}\left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,Fe_1)}\right)\max\left\{\varphi\rho(e_1,Fe_1),\varphi\rho(e_2,Te_2)\right\} &\geq \left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,1)}\right)\left[\tfrac{1}{4}\rho(e_1,1)\right]\\ &= \left(\tfrac{2e_1+e_2}{2e_1+2}\right)\left[\tfrac{1}{4}(2e_1+1)\right]\\ &\geq \tfrac{5}{6}\\ &\geq \tfrac{2}{4}.\end{aligned}$$

Subcase (2): If $e_2 \in \{6, 7, \cdots, 10\}$, then we get that

$$\varphi\rho(Fe_1, Te_2) = \varphi\rho(1, 1) = \varphi(3) = \frac{3}{4}.$$

$$\begin{aligned}\left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,Fe_1)}\right)\max\left\{\varphi\rho(e_1,Fe_1),\varphi\rho(e_2,Te_2)\right\} &= \left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,1)}\right)\left[\tfrac{1}{4}\rho(e_2,1)\right]\\ &= \left(\tfrac{2e_1+e_2}{2e_1+2}\right)\left[\tfrac{1}{4}(2e_1+1)\right]\\ &\geq \tfrac{13}{4}\left(\tfrac{2e_1+6}{2e_1+2}\right)\\ &\geq \tfrac{13}{4}\left(\tfrac{16}{12}\right)\\ &\geq \tfrac{3}{4}.\end{aligned}$$

Subcase (3): If $e_2 \in \{11, 12, \cdots, m\}$, then we get that

$$\varphi\rho(Fe_1, Te_2) = \varphi\rho(1, 2) = \varphi(4) = 1.$$

$$\begin{aligned}\left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,Fe_1)}\right)\max\left\{\varphi\rho(e_1,Fe_1),\varphi\rho(e_2,Te_2)\right\} &= \left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,1)}\right)\left[\tfrac{1}{4}\rho(e_2,2)\right]\\ &= \left(\tfrac{2e_1+e_2}{2e_1+2}\right)\left[\tfrac{1}{4}(2e_2+2)\right]\\ &\geq 6\left(\tfrac{2e_1+11}{2e_1+2}\right)\\ &\geq \tfrac{21}{2}\\ &\geq 1.\end{aligned}$$

Case (3): If $e_1 \in \{6, 7, \cdots, m\}$, then we have the following subcases:
Subcase (1): If $e_2 \in \{0, 1, \cdots, 5\}$, then we have

$$\varphi\rho(Fe_1, Te_2) = \varphi\rho(2, 0) = \varphi(4) = 1.$$

$$\begin{aligned}\left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,Fe_1)}\right)\max\left\{\varphi\rho(e_1,Fe_1),\varphi\rho(e_2,Te_2)\right\} &= \left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,2)}\right)\left[\tfrac{1}{4}\rho(e_1,2)\right]\\ &= \left(\tfrac{2e_1+e_2}{2e_1+3}\right)\left[\tfrac{1}{4}(2e_1+2)\right]\\ &\geq \left(\tfrac{2e_1}{2e_1+3}\right)\left[\tfrac{1}{4}(2e_1+2)\right]\\ &\geq \tfrac{14}{5}\\ &\geq 1.\end{aligned}$$

Subcase (2): If $e_2 \in \{6, 7, \cdots, 10\}$, then we have

$$\varphi\rho(Fe_1, Te_2) = \varphi\rho(2, 1) = \varphi(5) = \frac{5}{4}.$$

$$\begin{aligned}\left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,Fe_1)}\right)\max\left\{\varphi\rho(e_1,Fe_1),\varphi\rho(e_2,Te_2)\right\} &\geq \left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,2)}\right)\left[\tfrac{1}{4}\rho(e_1,2)\right]\\ &= \left(\tfrac{2e_1+e_2}{2e_1+3}\right)\left[\tfrac{1}{4}(2e_1+2)\right]\\ &\geq \left(\tfrac{2e_1+6}{2e_1+3}\right)\left[\tfrac{1}{4}(2e_1+2)\right]\\ &\geq \tfrac{21}{5}\\ &\geq \tfrac{5}{4}.\end{aligned}$$

Subcase (3): If $e_2 \in \{11, 12, \cdots, m\}$, then we get that

$$\varphi\rho(Fe_1, Te_2) = \varphi\rho(2, 2) = \varphi(6) = \frac{6}{4}.$$

$$\begin{aligned}\left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,Fe_1)}\right)\max\left\{\varphi\rho(e_1,Fe_1),\varphi\rho(e_2,Te_2)\right\} &\geq \left(\tfrac{\rho(e_1,e_2)}{1+\rho(e_1,2)}\right)\left[\tfrac{1}{4}\rho(e_2,2)\right]\\ &= \left(\tfrac{2e_1+e_2}{2e_1+3}\right)\left[\tfrac{1}{4}(2e_2+2)\right]\\ &\geq 6\left(\tfrac{2e_1+11}{2e_1+3}\right)\\ &\geq \tfrac{6}{4}.\end{aligned}$$

In a similar manner, we can show that:

$$\varphi\rho(Te_1, Fe_2) \leq \left(\frac{\rho(e_1, e_2)}{1 + \rho(e_1, Te_1)}\right)\max\left\{\varphi\rho(e_1, Te_1), \varphi\rho(e_2, Fe_2)\right\}.$$

Consequently, the pair (F, T) satisfies the conditions of Theorem 3 ensures that F and T have a unique common fixed point in E. □

3. APPLICATION

Theorem 7 *Let $m = 2^n$ with $n \in \mathbb{N}$. Then the function*

$$F(x) = \left[(1 - x^m)/(\eta - x^m)\right], \text{ where } \eta \geq m + 2$$

has a unique fixed point in $[0, 1]$.

Proof. Let $E = [0, 1]$. Define $q : E \times E \to [0, \infty)$ by $q(e_1, e_2) = |e_1 - e_2|$. Then (E, q) is a complete quasi metric space. Also, define $\rho : E \times E \to [0, \infty)$ by $\rho(e_1, e_2) = |e_1 - e_2|$. Then ρ is an mω-distance mapping. Now, equipped (E, q) with ρ.

Also, define $\varphi : [0, \infty) \to [0, \infty)$ by

$$\varphi(\mu_*) = \begin{cases} \mu_* & \text{if } \mu_* \in [0, 1];\\ (1/9)(\mu_*^2 + 1) & \text{if } \mu_* > 1.\end{cases}$$

Note that φ is an ultra distance function.
Now,

$$\begin{aligned}
\varphi\rho(Fe_1, Fe_2) &= \left|(\frac{1-e_1^m}{\eta - e_1^m}) - (\frac{1-e_2^m}{\eta - e_2^m})\right| \\
&= \left|\frac{(1-e_1^m)(\eta - e_2^m) - (1-e_2^m)(\eta - e_1^m)}{(\eta - e_1^m)(\eta - e_2^m)}\right| \\
&= \left(\frac{(\eta - 1)}{(\eta - e_1^m)(\eta - e_2^m)}\right)\left|e_1^m - e_2^m\right| \\
&= \left(\frac{(\eta - 1)}{\left.(\eta - e_1^m)\right)(\eta - e_2^m)}\right)\left[(e_1+e_2)(e_1^2+e_2^2)(e_1^4+e_2^4)\cdots(e_1^{\frac{m}{2}}+e_2^{\frac{m}{2}})\right]\left|e_1 - e_2\right| \\
&\leq \frac{(\eta - 1)(2^n)}{(\eta - 1)^2}\left|e_1 - e_2\right| \\
&= \frac{(\eta - 1)(m)}{(\eta - 1)^2}\left|e_1 - e_2\right| \\
&= \frac{(m)}{(\eta - 1)}\varphi\rho(e_1, e_2).
\end{aligned}$$

By taking $k = \frac{(m)}{(\eta-1)}$ then $k < 1$ and noting that F is continuous, we deduce that F satisfies all conditions of Corollary 6. Therefore, F has a unique fixed point in E. $\square$

Example 2 *The function*

$$F(x) = \left[(1 - x^{128})/(130 - x^{128})\right]$$

has a unique fixed point in $[0, 1]$.

Proof. By applying Theorem 7 with $m = 128$ and $\eta = 130$. $\square$

4. CONCLUSION

Based on the definition of modified ω-distance mappings, the notion of the $\epsilon_\phi-$contraction was introduced. By employ this new definition, we proved some fixed point result. An example was introduced to show the validity and reliability of our new results.

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