International Journal of Electrical and Computer Engineering (IJECE)
Vol. 10, No. 2, April 2020, pp. 1648∼1654
ISSN: 2088-8708, DOI: 10.11591/ijece.v10i2.pp1648-1654

Approximating offset curves using Bézier curves with high accuracy

Abedallah Rababah[1], Moath Jaradat[2]
[1]Department of Mathematical Sciences, United Arab Emirates University, United Arab Emirates
[1,2]Department of Mathematics, Jordan University of Science and Technology, Jordan

Article Info

Article history:

Received Apr 14, 2019
Revised Oct 20, 2019
Accepted Oct 30, 2019

Keywords:

Approximation order
Bézier curves
Circular arc
Cubic approximation
High accuracy

ABSTRACT

In this paper, a new method for the approximation of offset curves is presented using the idea of the parallel derivative curves. The best uniform approximation of degree 3 with order 6 is used to construct a method to find the approximation of the offset curves for Bézier curves. The proposed method is based on the best uniform approximation, and therefore; the proposed method for constructing the offset curves induces better outcomes than the existing methods.

Corresponding Author:

Abedallah Rababah,
Department of Mathematics,
Jordan University of Science and Technology,
22110 Irbid, Jordan.
Email: rababah@just.edu.jo and rababah@uaeu.ac.ae

1. INTRODUCTION

The offset curves appeared in the 19th century and are widely used in Computer Aided Design/Computer Aided Manufactoring CAD/CAM applications, and has other applications in many computer fields. Many studies on the offset approximation are carried out by many researchers. Hoschek [1] approximated the offset curves using splines. Rational offset curves are approximatedby Farouki and Sakkalis [2] by constructing the Pythagorean-hodograph (PH) curves. In [3], rational offset curves based on the quadratic approximation of the circular arc are approximated. Recently, offset approximation curves based on the circular arc approximations are presented [4-6] yielding rational offset approximation which are the convolution of the unit normal vector and the given curve. The offset approximation in this paper is based on the best uniform approximation of the circular arc and yields a polynomial offset approximation curve. The best uniform approximation of the circular arc of degree 3 presented in [7] where the error function is the Chebyshev polynomial of degree 6, see also [8-16]. .

This offset method is constructed as follows: given a Bézier curve $b(t)$ and its unit normal vector $N(t)$ which is a circular arc. Then we use the best uniform approximation of degree 3 to approximate the unit normal vector of the given curve. Since the best uniform approximation is of high accuracy then it is anticipated that the approximation of the normal vector is as of high accuracy. Thereafter, a special reparametrization of the approximation to unit normal vector $N^a(t)$ is carried out to have the same length as the unit normal vector $N(t)$. In this method one step approximation is used so the error will be less than other methods. There are three types of approximation with respect to the norm; L_1 norm, L_2 norm, and L_∞ norm which is the best uniform approximation that we are using in my paper. Cubic Bézier curves are commonly used in almost all industrial companies; it is used in computer graphics, animation, modeling, CAD, CAGD, design, and many other related fields. In these and other applications in CG and CAGD, conic sections are the most commonly used curves in any CAD system.

The Bernstein polynomials are one of the most important polynomials in mathematics. They serve essential tasks in numerical, approximation and Bézier curves, because they form basis which are numerically stable. The Bernstein basis polynomials of degree n are defined as [17-19]:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \qquad t \in [0,1], \quad i = 0, 1, 2, ..., n, \tag{1}$$

where the binomial coeffcients are given by

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

The Bernstein polynomials are used as basis for the approximation and representation of curves and are generalized to triangular surfaces [20,21]. The Bernstein polynomials are, in particular, important for the construction of the Bézier curves that are defined as follow.

A Bézier curve of degree n is defined by

$$b(t) = \sum_{i=0}^{n} b_i B_i^n(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad t \in [0,1], \tag{2}$$

where b_i's are the control points, and $B_i^n(t)$ are the Bernstein polynomials of degree n.

For a given Bézier curve $b(t)$ in (2), the offset curve $b_r(t)$ with offset distance $r \in \mathbb{R}^+$ is given by

$$b_r(t) = b(t) + rN(t), \tag{3}$$

where $N(t)$ is the unit normal vector of $b(t)$ given by

$$N(t) = \frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}. \tag{4}$$

The error function $e(t)$ is used to measure the error between $N(t)$ and $N^a(t)$ and is given by

$$e(t) = (\frac{y'(t)}{\sqrt{x'^2(t) + y'^2(t)}} - \frac{y'_a(t)}{\sqrt{x'^2_a(t) + y'^2_a(t)}})^2 + (\frac{x'(t)}{\sqrt{x'^2(t) + y'^2(t)}} - \frac{x'_a(t)}{\sqrt{x'^2_a(t) + y'^2_a(t)}})^2$$

2. RESEARCH METHOD

In this section, we present a new method of offset curve approximation of the n-th degree Bézier curve by a curve of degree 3. The best uniform approximation of the circular arc of degree 3 of order 6 is presented in [7], see also [22-24]. The cubic approximation of circular arc $p(t)$ has a parametrically defined polynomial curve given by

$$p(t) = \begin{pmatrix} -0.515647 + 5.99959t - 5.99959t^2 \\ -0.874847 - 2.25031t + 12t^2 - 8t^3 \end{pmatrix}, \qquad t \in [0,1]. \tag{5}$$

Let $b(t)$ be a regular planar Bézier curve of degree n given in (2) and $N(t)$ be its unit normal vector given in (4). As shown in Figure 1., given any Bézier curve $b(t)$ then by the definition of the convolutio, the tangent line of $b(t)$ is parallel to the tangent line of $N(t)$ which is the unit normal vector for $b(t)$, $\forall\, t \in [0,1]$.

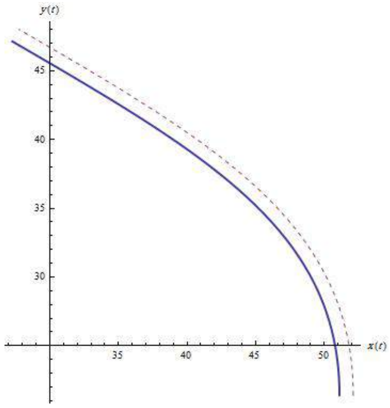

Figure 1. Tangent of $b(t)$ (thick) parallel to the tangent of $N(t)$ (dashed)

Thus

$$b * rN(t) = b(t) + rN(t) = b_r(t).$$

Since $N(t)$ is circular arc, the tangent line of $N(t)$ is parallel to the tangent line of $b(t)$, then the approximation of $N(t)$ is also circular arc and parallel to $b(t)$. Note that, $N^a(s(t))$ and $b(t)$ have the same unit normal vector. So, the offset approximation is given by

$$b_r^a(t) = b * rN^a(s(t)) = b(t) + rN^a(s(t)), \quad t \in [0,1]. \tag{6}$$

The construction of the approximation $N^a(s(t))$, $t \in [0,1]$ for the cubic Bézeir curve is considered.

$$N^a(s) = \left(\frac{(-2.25031 + 24s - 24s^2), -(5.99959 - 11.9992s)}{\sqrt{(5.99959 - 11.9992s)^2 + (-2.25031 + 24s - 24s^2)^2}} \right), \tag{7}$$

where $s = s(t)$, $t \in [0,1]$ is regular reparametrization to make both curves begin and end at the same points. The curve defined by

$$b_r^a(t) = b * N^a(s(t)) = b(t) + rN^a(s(t)), \quad t \in [0,1],$$

is the approximation of the offset curve by cubic Bézier curve where $N^a(s(t))$ is as in (7).

The computation of the reparametrization, $s = s(t)$, where $t \in [0,1]$ is considered. $N^a(t)$ and $N(t)$ have different parameters, both of them are circular arcs, but they do not have the same start and end points. Figure 2. shows $N^a(t)$ and $N(t)$ for a Bézier curve $b(t)$. A reparametrization $s = s(t)$ is presented, so that the curve and its approximation begin and end at the same points.

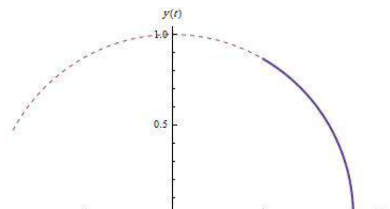

Figure 2. $N(t)$ (thick) for Bézier curve and the approximation $N^a(t)$ (dashed)

Let $b(t)$ be the curve in (2) and $N(t)$ be its unit normal vector. The reparametrization is

$$s(t) = \frac{t}{b} + \frac{1-t}{a}, \quad t \in [0,1],$$

where a, b are given by the follwing: the first step we find $N(0)$ and $N(1)$ for the given curve, $b(t)$. We get to solve the equations

$$N^a(\frac{1}{a}) = \begin{pmatrix} x^a(\frac{1}{a}) \\ y^a(\frac{1}{a}) \end{pmatrix} = N(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \tag{8}$$

$$N^a(\frac{1}{b}) = \begin{pmatrix} x^a(\frac{1}{b}) \\ y^a(\frac{1}{b}) \end{pmatrix} = N(1) = \begin{pmatrix} x(1) \\ y(1) \end{pmatrix}. \tag{9}$$

By the symmetry of the approximation of the circular arc, in equation (8), $x^a(\frac{1}{a})$ and $y^a(\frac{1}{a})$ equal to zero at the same parameters, and (9), $x^a(\frac{1}{b})$ and $y^a(\frac{1}{b})$ equal to one at the same parameters, then a equals the parameter in (8) and b equals the parameter in (9).

By solving the following equations

$$\frac{(-2.25031 + 24(\frac{1}{a}) - 24(\frac{1}{a})^2)}{\sqrt{(5.99959 - 11.9992(\frac{1}{a}))^2 + (-2.25031 + 24(\frac{1}{a}) - 24(\frac{1}{a})^2)^2}} = x(0)$$

and

$$\frac{-(5.99959 - 11.9992(\frac{1}{a}))}{\sqrt{(5.99959 - 11.9992(\frac{1}{a}))^2 + (-2.25031 + 24(\frac{1}{a}) - 24(\frac{1}{a})^2)^2}} = y(0)$$

we get the value of the parameter a.

And by solving

$$\frac{(-2.25031 + 24(\frac{1}{b}) - 24(\frac{1}{b})^2)}{\sqrt{(5.99959 - 11.9992(\frac{1}{b}))^2 + (-2.25031 + 24(\frac{1}{b}) - 24(\frac{1}{b})^2)^2}} = x(1)$$

and

$$\frac{-(5.99959 - 11.9992(\frac{1}{b}))}{\sqrt{(5.99959 - 11.9992(\frac{1}{b}))^2 + (-2.25031 + 24(\frac{1}{b}) - 24(\frac{1}{b})^2)^2}} = y(1)$$

we get the value of the parameter b.

Then the approximation of $N^a(t)$ for the cubic case is

$$N^a(t) = \left(\frac{(-2.25031 + 24(\frac{t}{b} + \frac{1-t}{a}) - 24(\frac{t}{b} + \frac{1-t}{a})^2), -(5.99959 - 11.9992(\frac{t}{b} + \frac{1-t}{a}))}{\sqrt{(5.99959 - 11.9992(\frac{t}{b} + \frac{1-t}{a}))^2 + (-2.25031 + 24(\frac{t}{b} + \frac{1-t}{a}) - 24(\frac{t}{b} + \frac{1-t}{a})^2)^2}} \right).$$

3. RESULTS AND ANALYSIS

The method is applied for the following cubic parametric curve:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 27.2688t^3 + 341.56752t^2(1-t) + 351.1t(1-t)^2 + 51.1(1-t)^3 \\ 47.1461t^3 + 338.8523975t^2(1-t) + 333.324t(1-t)^2 + 21.4(1-t)^3 \end{pmatrix}, \tag{10}$$

where $t \in [0,1]$ and the unit normal vector is given by:

$$N(t) = \begin{pmatrix} \frac{(35.772(1-t)^2 + 33.1704(1-t)t + 24.8811t^2)}{\sqrt{(35.772(1-t)^2 + 33.1704(1-t)t + 24.8811t^2)^2 + (0.(1-t)^2 - 57.1949(1-t)t - 42.8962t^2)^2}} \\ \frac{(-(0.(1-t)^2 - 57.1949(1-t)t - 42.8962t^2))}{\sqrt{(35.772(1-t)^2 + 33.1704(1-t)t + 24.8811t^2)^2 + (0.(1-t)^2 - 57.1949(1-t)t - 42.8962t^2)^2}} \end{pmatrix}.$$

Figure 3. represents the graph of the cubic parametric curve and Figure 4. is the cubic parametric curve with its offset curve computed by the formula. Figure 5. illustrates the parametric cubic curve with the cubic approximation of the offset curve and the original offset curve. And Figure 6. illustrates the error between the offset curve and the approximation of the offset curve.

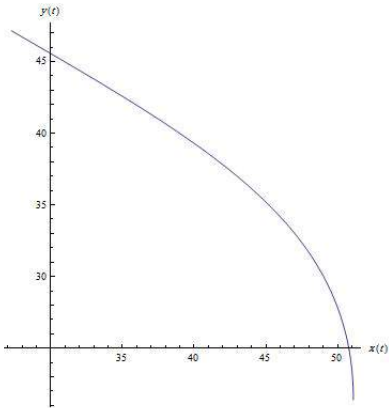

Figure 3. The cubic parametric curve

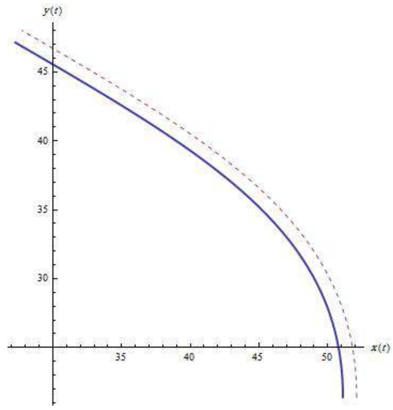

Figure 4. Cubic parameteric curve (thick) and its offset curve (dashed)

By solving eqautions (8) and (9), we get

$$a = 2, \qquad b = 1.28862.$$

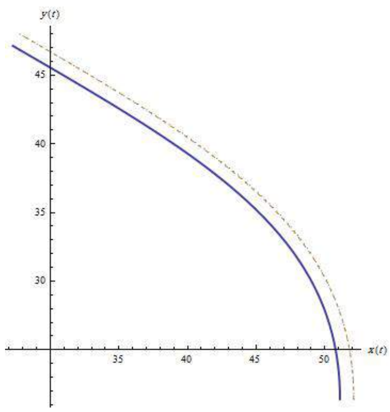

Figure 5. Cubic curve (thick) and its offset curve (dashed) and the cubic approximation of the offset curve (dotted)

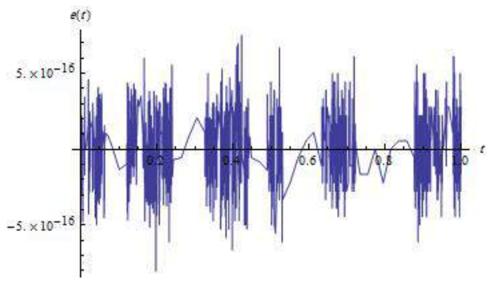

Figure 6. Error between offset curve and the approximation offset curve

4. CONCLUSION

In this article, cubic approximation of offset curve is established. The method is based on the best uniform approximation of the circular arc of degree 3 with order 6. The numerical examples reveal how efficient this method is. The maximum error is 5×10^{-16}, thus the proposed method induced better outcomes than the existing methods. The results in this paper can be used to improve the results obtained in [25], see also the results in [26].

ACKNOWLEDGEMENTS

The authors owe thanks for the reviewers for invaluable suggestions to improve an earlier version of this paper. This research was funded by Jordan University of Science and Technology.

REFERENCES

[1] Josef Hoschek. ”Spline approximation of offset curves.” Computer Aided Geometric Design vol. 5, Issue 1, pp. 33-40, 1988.

[2] R. Farouki and Takis Sakkalis. ”Pythagorean hodographs.” IBM Journal of Research and Development vol. 34, Issue 5, pp. 736-752, 1990.

[3] Lee, In-Kwon, Myung-Soo Kim, and Gershon Elber. ”Planar curve offset based on circle approximation.” Computer-Aided Design vol. 28, Issue 8, pp. 617-630, 1996.

[4] Ahn, Young Joon, Yeon Soo Kim, and Youngsuk Shin. ”Approximation of circular arcs and offset curves by Bézier curves of high degree.” Journal of Computational and Applied Mathematics vol. 167, Issue 2 , pp. 405-416, 2004.

[5] Kim, Soo Won, Sung Chul Bae, and Young Joon Ahn. ”An algorithm for G2 offset approximation based on circle approximation by G2 quadratic spline.” Computer-Aided Design vol. 73 , pp. 36-40, 2016.

[6] Kim, Soo Won, Ryeong Lee, and Young Joon Ahn. ”A new method approximating offset curve by Bézier curve using parallel derivative curves.” Computational and Applied Mathematics vol. 37, Issue 2, pp. 2053-2064, 2018.

[7] A. Rababah, ”The best uniform cubic approximation of circular arcs with high accuracy.” Communications in Mathematics and Applications vol. 7, Issue 1, pp. 37-46, 2016.

[8] A. Rababah, The best uniform quadratic approximation of circular arcs with high accuracy, Open Mathematics 14 (1), 118-127, 2016.

[9] A. Rababah, Quartic approximation of circular arcs using equioscillating error function, Int. J. Adv. Comput. Sci. Appl 7 (7), 590-595, 2016.

[10] Takashi Maekawa, An overview of offset curves and surfaces, Computer-Aided Design, Volume 31, Issue 3, 1999, Pages 165-173. https://doi.org/10.1016/S0010-4485(99)00013-5

[11] B. Pham, Offset curves and surfaces: a brief survey, Computer-Aided Design Volume 24, Issue 4, 1992, Pages 223-229. https://doi.org/10.1016/0010-4485(92)90059-J

[12] Huanxin Cao, Gang Hu, Guo Wei, and Suxia Zhang, Offset Approximation of Hybrid Hyperbolic Polynomial Curves, Results in Mathematics, 2017, Volume 72, Issue 3, 1055-1071.

[13] Bhargav Bhatkalkar, Abhishek Joshi, Srikanth Prabhu, Sulatha Bhandary, Automated fundus image quality assessment and segmentation of optic disc using convolutional neural networks. International Journal of Electrical and Computer Engineering (IJECE) Vol.10, No.1, 2020, pp. 816 827, ISSN: 2088-8708, DOI: 10.11591/ijece.v10i1.pp816-827.

[14] A. Rababah, The best quintic Chebyshev approximation of circular arcs of order ten, International Journal of Electrical and Computer Engineering (IJECE)Vol. 9, No. 5, October 2019, pp. 3779-3785, ISSN: 2088-8708, DOI: 10.11591/ijece.v9i5.pp3779-3785.

[15] Rodiah, Sarifuddin Madenda, Diana Tri Susetianigtias, Dewi Agushinta Rahayu, EtySutanty, Optic Disc and Macula Localization from Retinal Optical Coherence Tomography and Fundus Image, International Journal of Electrical andComputer Engineering (IJECE) Vol.8, No.6, 2018, pp. 5050-5060, ISSN: 2088-8708, DOI: 10.11591/ijece.v8i6.pp5050-5060.

[16] G. Daway, Hana H. kareem, Ahmed Rafid Hashim, Pupil Detection Based on Color Difference and Circular Hough TransformHazim, International Journal of Electrical and Computer Engineering (IJECE) Vol.8, No.5, 2018, pp. 3278-3284, ISSN: 2088-8708, DOI: 10.11591/ijece.v8i5.pp3278-3284.

[17] Gerald Farin, Curves and surfaces for computer-aided geometric design: a practical guide. Elsevier, 2014.

[18] K. Höllig and J. Hörner, ”Approximation and Modeling with B-Splines,” *SIAM, Titles in Applied Mathematics*, vol. 132, 2013.

[19] H. Prautzsch., et al., ”Bézier and B-Spline Techniques,” Springer 2002.

[20] A. Rababah, L-2 degree reduction of triangular Bézier surfaces with common tangent planes at vertices. International Journal of Computational Geometry & Applications, Vol. 15, No. 5 (2005) 477-490. http://www.worldscinet.com/ijcga/15/1505/S02181959051505.html

[21] A. Rababah,, Distances with rational triangular Bézier surfaces. Applied Mathematics and Computation 160, (2005) 379-386. http://www.elsevier.com/locate/amc

[22] Tor Dokken, et al. ”Good approximation of circles by curvature-continuous Bézier curves.” Computer Aided Geometric Design vol. 7, Issue 1-4, pp. 33-41, 1990.

[23] Elber Gershon, In-Kwon Lee, and Myung-Soo Kim. ”Comparing offset curve approximation methods.” IEEE computer graphics and applications vol. 17, Issue 3, pp. 62-71, 1997.

[24] A. Rababah, "Taylor theorem for planar curves," Proceedings of the American Mathematical Society, vol. 119, Issue 3, pp. 803–810, 1993.
[25] C. Suwannapong and C. Khunboa, "An Approximation Delay between Consecutive Requests for Congestion Control in Unicast CoAP-based Group Communication," *International Journal of Electrical and Computer Engineering (IJECE)*, vol. 9, Issue 3, June 2019. DOI: http://doi.org/10.11591/ijece.v9i3.
[26] A. Rababah, "Approximation von Kurven mit Polynomen und Splines," *Ph. Dissertation, Stuttgart Universität*, West Germany, 1992.

BIOGRAPHIES OF AUTHORS

Abedallah Rababah is a professor of mathematics at United Arab Emirates University and is on leave from Jordan University of Science and Technology. He is working in the field of Computer Aided Geometric Design, abbreviated CAGD. In particular, his research is on degree raising and reduction of Bézier curves and surfaces with geometric boundary conditions, Bernstein polynomials, and their duality. He is known for his research in describing approximation methods that significantly improve the standard rates obtained by classical (local Taylor, Hermite) methods. He proved the following conjecture for a particular set of curves of nonzero measure: Conjecture: A smooth regular planar curve can, in general, be approximated by a polynomial curve of degree n with order 2n. The method exploited the freedom in the choice of the parametrization and achieved the order 4n/3, rather than n + 1. Generalizations were also proved for space curves. Professor Rababah is also doing research in the fields of classical approximation theory, orthogonal polynomials, Jacobi-weighted orthogonal polynomials on triangular domains, and best uniform approximations. Since 1992, He has been teaching at German, Jordanian, American, Canadian, and Emirates' universities. He is active in the editorial boards of many journals in mathematics and computer science. Further info can be found on his homepage at ResearchGate:
https://www.researchgate.net/profile/Abedallah_Rababah
or at http://www.just.edu.jo/eportfolio/Pages/Default.aspx?email=rababah

Moath Jaradat graduated from Jordan University of Science and Technology. He obtained both Bachelor and Master Degrees in Mathematics with research interests in the field of approximating offset curves using Bézier curves with high accuracy and numerical and approximations. Further info can be found on his homepage at ResearchGate:
https://www.researchgate.net/profile/Moad_Jaradat