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Computational Sinc-scheme for extracting analytical solution for the model Kuramoto-Sivashinsky equation

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ABSTRACT

The present article is designed to supply two different numerical solutions for solving Kuramoto-Sivashinsky equation. We have made an attempt to develop a numerical solution via the use of Sinc-Galerkin method for Kuramoto-Sivashinsky equation, Sinc approximations to both derivatives and indefinite integrals reduce the solution to an explicit system of algebraic equations. The fixed point theory is used to prove the convergence of the proposed methods. For comparison purposes, a combination of a Crank-Nicolson formula in the time direction, with the Sinc-collocation in the space direction is presented, where the derivatives in the space variable are replaced by the necessary matrices to produce a system of algebraic equations. In addition, we present numerical examples and comparisons to support the validity of these proposed methods.

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1. INTRODUCTION

The Kuramoto-Sivashinsky equation (that we abbreviate as K-S), is a simple one-dimensional partial differential equation that exhibiting a particular complex dynamical behavior under some conditions. It arises as an amplitude equation in long-wave, weakly nonlinear stability in a great variety of applications. For example, it arises in concentration waves in chemically reacting systems [1], in flame propagation and reaction combustion [2]. In its simplest form, the equation is given by subject to the initial condition

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxxx} = 0,\ x \in \mathbb{R},\ t > 0 \tag{1}$$

$$u(x,0) = f(x),\ x \in \mathbb{R} \tag{2}$$

We seek a real-valued function $u = u(x,t)$, defined on $\mathbb{R} \times \mathbb{R}_0^+$ satisfying (1) with $f : \mathbb{R} \to \mathbb{R}$ is sufficiently smooth function satisfying some decay conditions. The u_{xx} term in (1) is responsible for an instability at large scales, where α is a positive constant; the dissipative u_{xxxx} term provides damping at small scales, and the positive constant β playing the role of viscosity; and the non-linear term uu_x (which has the same form as that in the Burgers or one-dimensional Navier Stokes equations) stabilizes by transferring energy between large and small scales. The K-S equation, where it models cellular instabilities, pattern formation, turbulence phenomena and transition to chaos. A complete account of the numerical literature on these approximations are mentioned in [3, 4, 5, 6, 7, 8]. Mesh-free methods are the topic of recent research in many areas

of computational science and approximation theory [9]. Over the past several years mesh-free approximation methods have found their way into many different application areas ranging from engineering applications to the numerical solution of differential equations. A meshless method does not require grid, and only makes use of a set of scattered collocation points. In [10, 11, 12], the authors proposed a mesh-free collocation method and formulate a simple classical radial basis functions for the numerical solution of the KdV equation, coupled KdV equations and the K-S equation.

2. RELATED WORKS

In recent years various methods have been presented to find approximate solutions for K-S equation. In [13], the quintic B-spline collocation method is implemented to find numerical solution of the Kuramoto-Sivashinsky equation. In [14] a numerical technique based on the finite difference and collocation methods is presented for the generalized Kuramoto-Sivashinsky equation. Chebyshev spectral collocation methods are used in [15] to find approximate solutions for the generalized K-S equation. In [16], authors analyzed and implemented a fully discrete schemes for periodic initial value problems for a general class of dispersively modified Kuramoto-Sivashinsky equations. Time discretizations are constructed using linearly implicit schemes and spectral methods are used for the spatial discretization.

3. RESEARCH METHOD

In this paper, we approximate the solution of the K-S equation (1) subject to the initial condition (2) using Sinc-Galerkin method, which builds an approximate solution valid on the entire spatial domain and on a small interval in the time domain. Another benefit of the Sinc methodology is that the scheme presented automatically handles singularities occur at the boundaries with ease. For more details about Sinc solutions of analytic problems with singularties, see [17]. The main idea is to replace derivatives and integrals by their discrete Sinc approximation. Also we solve the K-S solution (1) in the region $a \leq x \leq b,\ t \geq 0$, by discretizing in time through the use of Crank-Nicolson scheme, and in space by Sinc-collocation method. The layout of the paper is as follows: In section 2, we give the relevant properties of Sinc function such as notations, definitions and some theorem that we need. In section 3, we developed the Sinc-Galerkin method for solving K-S equations, and a convergence proof is also given. In section 4, we discuss the mesh-free method together with the Sinc-collocation discretization of the K-S equation. Finally, numerical experiments are presented and some comparisons are made in section 5. Some concluding remarks are given in the final section.

4. PROPOSED METHOD

The goal of this section is to recall notations and definitions of the Sinc function that will be used in this paper. These are discussed in [3, 4], and mainly, we will recall section 2 of [6, 18, 19]. The Sinc function is defined on the whole real line $\mathbb{R}$ by

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, x \in \mathbb{R}. \tag{3}$$

4.1. Preliminaries

Recall that a radial basis function is a function whose value depends only on the distance of its input to a central point. For a series of nodes equally spaced h apart, the Sinc function can be written as a radial basis function:

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} \tag{4}$$

Let $\delta_{kj}^{(-1)} = \frac{1}{2} + \delta_{kj}$, where

$$\delta_{kj} = \int_0^{k-j} \frac{\sin(\pi t)}{\pi t} dt.$$

We define a matrix $I^{(-1)}$ whose $(k,j)th$ entry is given by $\delta_{kj}^{(-1)}$. If a function $f(x)$ is defined on the real line, then for $h > 0$, the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)\text{sinc}\left(\frac{x-jh}{h}\right), j = 0, \mp 1, ...$$

is called the Whittaker cardinal expansion, which has been extensively studies in [4, 3]. In practice, we need to use a finite number of terms in the above series, say $j = -N, ..., N$, where N is the number of sinc grid points. For a restricted class of functions known as the Paly-Weiner class, which are entire functions, the sinc interpolation and quadrature formulae are exact [4]. A less restricted class of functions that are analytic only on an infinite strip containing the real line, and that allow specific growth restrictions has exponentially decaying absolute errors in the sinc approximation.

Definition 1 *Let $\mathcal{D}_d$ denote the infinite strip domain of width $2d,\ d > 0$, given by*

$$\mathcal{D}_d = \{w = u + iv : |v| < d \leq \pi/2\}$$

To construct approximation on the interval $\Gamma = (0, T_0)$, which is our time interval in this paper, we consider the conformal map $\Upsilon(t) = \ln(\frac{t}{T_0 - t})$, the map Υ carries the eye-shaped region

$$\mathcal{D} = \Big\{z = x + iy : \Big|\arg\Big(\frac{z}{T_0 - z}\Big)\Big| < d \leq \pi/2\Big\}.$$

onto the infinite strip $\mathcal{D}_d$. For the sinc method, the basis functions on the interval Γ at $z \in \mathcal{D}$ are derived from the composite translated sinc functions

$$S_j(z) = S(j,h) \circ \Upsilon(z) = \text{sinc}\left(\frac{\Upsilon(z) - jh}{h}\right).$$

The function $z = \Upsilon^{-1}(w) = \frac{T_0 \exp(w)}{1+\exp(w)}$ is an inverse mapping of the $w = \Upsilon$. We define the range of Υ^{-1} on the real line as

$$\Gamma = \{\Upsilon^{-1}(y) \in \mathcal{D} : -\infty < y < \infty\} = (0, T_0).$$

The sinc grid points $z_k \in \Gamma$ in $\mathcal{D}$ will be denoted by t_k, because they are real, and is given by

$$t_k = \Upsilon^{-1}(kh) = \frac{T_0 \exp(kh)}{1 + \exp(kh)},\ k = 0, \mp 1, \mp 2, ...$$

To construct an approximation in the interval $(-\infty, \infty)$, which is our space domain in this part, we replace Υ by $\phi(x) = x$. To further explain of the sinc method, an important class of functions is denoted by $\mathbf{L}_\alpha(\mathcal{D})$. The properties of the functions in $\mathbf{L}_\alpha(\mathcal{D})$ and detailed discussion are given in [4]. We recall the following definition followed by two Theorems for our purpose.

Definition 2 *Let $\mathbf{L}_\alpha(\mathcal{D})$ be the class of all analytic functions f in $\mathcal{D}$, for which there is a number C_0 such that, for $\rho(z) = \exp(\Upsilon(z))$, we have*

$$|f(z)| \leq C_0 \frac{|\rho(z)|^\alpha}{[1 + |\rho(z)|]^{2\alpha}},\ \forall z \in \mathcal{D}.$$

If x is on the arc Γ, we obtain the following theorem

Theorem 1 *Let $f(x) \in \mathbf{L}_\alpha(\mathcal{D})$, α a positive constant, then taking $h_x = \sqrt{\pi d/(\alpha N_x)}$ it follows that*

$$\sup_{x \in \Gamma} \Big| f^{(n)}(x) - \sum_{j=-N_x}^{N_x} f(x_j) S_j^{(n)}(x) \Big| \leq C_1 N_x^{\frac{n+1}{2}} \exp(-\sqrt{\pi d \alpha N_x})$$

for $n = 0, 1, ..., m$ with C_1 a constant independent of N_x.

In the next Theorem, we shall give a general formula for approximating the integral $\int_a^\nu F(u)du, \nu \in \Gamma$. To this end, we state the following result, which we will use to approximate the obtained integral equation.

Theorem 2 *Let* $\frac{F(t)}{\Upsilon'(t)} \in \mathbf{L}_\alpha(\mathcal{D})$*, with* $0 < \alpha \leq 1$*,* $\delta_{jk}^{(-1)}$ *be defined as above,* N_t *be positive integer, and* h_t *be selected as* $h_t = \sqrt{\pi d/(\alpha N_t)}$*, then there exists a positive constant* C_2 *independent of* N_t*, such that*

$$\Big| \int_a^{t_k} F(t)dt - h_t \sum_{j=-N_t}^{N_t} \delta_{jk}^{(-1)} \frac{F(t_k)}{\Upsilon'(t_k)} \Big| \leq C_2 \exp(-\sqrt{\pi d \alpha N_t})$$

The sinc method requires that the derivatives of sinc functions be evaluated at the nodes. Technical calculations provide the following results that will be useful in formulating the discrete system [4, 3], and these quantities are delineated by

$$\delta_{jk}^{(m)} = h^m \tfrac{d^m}{dx^m}[S_j \circ \phi(x)]\Big|_{x=x_k}, \text{ where}$$

$$\delta_{jk}^{(0)} = \begin{cases} 1, & j = k \\ 0, & j \neq k, \end{cases} \quad \delta_{jk}^{(2)} = \begin{cases} \frac{-\pi^2}{3}, & j = k \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k \end{cases} \quad \text{and,} \quad \delta_{jk}^{(4)} = \begin{cases} \frac{\pi^4}{5}, & j = k \\ \frac{4(-1)^{k-j}[6-(k-j-1)^2\pi^2]}{(k-j-1)^4}, & j \neq k \end{cases}$$

Then we define the $m \times m$ Toeplitz matrices $I^{(q)}, q = 0, 1, 2, 4$ whose $jk - th$ entry is $\delta_{jk}^{(q)}, q = 0, 1, 2, 4$. The matrix $I^{(0)}$ is the identity matrix.

Also we define the diagonal matrix $\mathcal{D}(g(x)) = \text{diag}\,[g(x_{-N}), ..., g(x_0), ..., g(x_N)]^T$.

4.2. Implementation of the method

The objective of this section is to construct a solution to the K-S equation using the Sinc-Galerkin method. Integrate equation (1) in section 2 with respect to t, we get the Volterra integral equation

$$u(x,t) = -\int_0^t \Big(u(x,\tau)u_x(x,\tau) + \alpha u_{xx}(x,\tau) + \beta u_{xxxx}(x,\tau)\Big)d\tau + f(x) \tag{5}$$

with the assumption that the initial condition $f(x) \in \mathbf{L}_\alpha(\mathcal{D})$. To obtain direct discretization of equation (5), and since our domain is $(x,t) \in \mathbb{R} \times (0, T_0)$, the relevant maps are defined as follows. In the space direction, we choose $\phi(x) = x$, which maps the infinite strip $\mathcal{D}_d$ onto itself. In the time direction, we choose the map $\Upsilon(t) = \ln(t/(T_0 - t))$ that carries the region $\mathcal{D}$ onto $\mathcal{D}_d$. Define the basis elements for $(-\infty, \infty), (0, T_0)$ to be $S(m, h_x) \circ \phi(x), m = -N_x, ..., N_x$, $S(k, h_t) \circ \Upsilon(t), k = -N_t, ..., N_t$, respectively. The mesh sizes h_x and h_t represent the mesh sizes in the infinite strip $\mathcal{D}_d$ for the uniform grid $\{ih_x\}, -\infty < i < \infty$ and $\{jh_t\}, -\infty < j < \infty$. The Sinc grid points $x_i \in (-\infty, \infty)$ in $\mathcal{D}_d$ and $t_j \in (0, T_0)$ in $\mathcal{D}$ are inverse images of equispaced grid points, i.e., $x_i = \phi^{-1}(ih_x) = ih_x$ and $t_j = \Upsilon^{-1}(jh_t) = \frac{T_0 \exp(jh_t)}{1+\exp(jh_t)}$. In equation (5), we carry out Sinc approximations of $u_x(x,t), u_{xx}(x,t)$ and $u_{xxxx}(x,t)$; to proceed, we use Theorem 1, and the replacement of the derivatives with respect to x by its approximation yields

$$u_x(x,t) \approx \Big(\frac{-1}{h_x} I_{m_x}^{(1)}\Big)u(x_i,t), \quad u_{xx}(x,t) \approx \Big(\frac{1}{h_x^2} I_{m_x}^{(2)}\Big)u(x_i,t), \quad \text{and} \;\; u_{xxxx}(x,t) \approx \Big(\frac{1}{h_x^4} I_{m_x}^{(4)}\Big)u(x_i,t).$$

where $m_x = 2N_x + 1$. In equation (5), evaluating all functions at the $x-$nodes, and replacing the derivative by its appropriate approximation, we obtain the Volterra integral equation

$$\mathbf{u}(t) = -\int_0^t \Big(\mathbf{u}(\tau)A_1\mathbf{u}(\tau) + \alpha A_2\mathbf{u}(\tau) + \beta A_4\mathbf{u}(\tau)\Big)d\tau + \mathbf{f}^0 \tag{6}$$

where $A_1 = \frac{-1}{h_x} I_{m_x}^{(1)}$, $A_2 = \frac{1}{h_x^2} I_{m_x}^{(2)}$, and

$$A_4 = \frac{1}{h_x^4} I_{m_x}^{(4)} \tag{7}$$

with $\mathbf{u}(t) = (u_{-N_x}, ..., u_{N_x})^T$, where $u_i(t) = u(x_i, t)$ and $\mathbf{f}^0 = (f(z_{-N_x}), ..., f(x_{N_x})$. We next collocate with respect to the t variable via the use of Theorem 2, with the matrix $B = h_t I_{m_t}^{(-1)} \mathcal{D}\Big(\frac{1}{\Upsilon'}\Big)$, $m_t = 2N_t + 1$ and the nodes $t_j = \Upsilon^{-1}(jh_t)$ for $j = -N_t, ..., N_t$. Define the matrix $F^0 = [f(x_i, 0)]$. Then the solution of equation (6) in matrix form is given by the rectangular $m_x \times m_t$ matrix $U = [u_{ij}]$,

$$U = -\Big(U \circ A_1 U + \alpha A_2 U + \beta A_4 U\Big) B^T + F^0 \tag{8}$$

where the notation $\circ$ denotes the Hadamard matrix multiplication. Note that in our discretization, we are taking the time nodes as rows and the space nodes as columns, so the matrix $\Big(U \circ A_1 U + \alpha A_2 U + \beta A_4 U\Big)$ forms the vector nodes for the integral in (6). In equation (8) the vector F^0 has the same dimensions as the vector U, and every column of F^0 consists of the same vector f^0.

To solve the system in equation (8), one idea is to produce a sequence of iterations that converges to the exact solution of the K-S equation. Equation (8) can be written as

$$U = G(U) + F^0 \tag{9}$$

where $G(U) = -\Big(U \circ A_1 U + \alpha A_2 U + \beta A_4 U\Big) B^T$. By selecting an initial approximation U^0, we iterate the continuous map G repeatedly via the formula

$$U^{(n+1)} = G(U^{(n)}) + F^0, \ n = 0, 1, 2, ...$$

4.3. Validation of Sinc-Galerkin method

Throughout this subsection, we use the notation $U = [u(x_i, t_j)]$ to denote the $m_x \times m_t$ matrix of node values of the function $u(x, t)$, and so on for other functions. Set, $K(x, t) = -\int_0^t M(\tau) d\tau$, where $M(\tau) = \Big(u(x, \tau) u_x(x, \tau) + \alpha u_{xx}(x, \tau) + \beta u_{xxxx}(x, \tau)\Big)$. Using Theorem 2, the approximation of the integral in matrix form has an error as

$$\| \ [K(x_i, t_j)] - BU \ \| \leq C_2 \exp(-\sqrt{\pi d \alpha N_t}) \tag{10}$$

In the next Theorem, we show that our approximation produce an error of exponential order

Theorem 3 *Let $u(x, t)$ be the exact solution of the $K - S$ equation, and let U be the matrix defined as in (9). Then for $N_x, N_t > \frac{16}{\pi d \alpha}$, there is a constant C independent of N_x, N_t such that*

$$\sup_{(x_i, t_j)} \| \ u(x, t) - U \ \| \leq C N^2 \exp(-\sqrt{\pi d \alpha N}), \quad \textit{where} \quad N = \min\{N_x, N_t\}$$

.

Proof: Evaluate the integral equation (5) at the nodes (x_i, t_j) where $i = -N_x, ..., N_x, \ j = -N_t, ..., N_t$, we get

$$u(x_i, t_j) = -\int_0^{t_j} \Big(u(x_i, \tau) u_x(x_i, \tau) + \alpha u_{xx}(x_i, \tau) + \beta u_{xxxx}(x_i, \tau)\Big) d\tau + f(x_i) \tag{11}$$

To approximate the above integral, we use the definite integral formula Theorem 2, and define a matrix $B = h_t I_{m_t}^{(-1)} \mathcal{D}\Big(\frac{1}{\Upsilon'}\Big)$, with $m_t = 2N_t + 1$, we obtain

$$U = -\Big(U \circ U_x + \alpha U_{xx} + \beta U_{xxxx}\Big) B^T + F^0 + C_2 \exp(-\sqrt{\pi d \alpha N_t}). \tag{12}$$

With the use of the approximations mentioned in Theorem 1, we obtain

$$\text{Error} = U + \Big(U \circ A_1 U + \alpha A_2 U + \beta A_4 U\Big) B^T - F^0 = C_2 \exp(-\sqrt{\pi d \alpha N_t}) + C_{12} N_x^2 B^T \exp(-\sqrt{\pi d \alpha N_x}).$$

But from the definition of the matrix B, and since $\frac{1}{\Upsilon'(t)} = \frac{t(T_0-t)}{T_0}$, which has its maximum at $T_0/4$, therefore, $B = T\tilde{B}$. It is known that $\| I_{m_t}^{(-1)} \| \leq 1.1$, (see, [4]), so the matrix B can be written as $B = T_0\tilde{B}$ where each entry in the matrix $\tilde{B}$ is bounded by $1.1h_tT_0/4$. With all of these bounds, the error term can be bounded as

$$\text{Error} = U - \Big(- U \circ A_1U - \alpha A_2U - \beta A_4U\Big)B^T - F^0 \leq CN^2 \exp(-\sqrt{\pi d\alpha N})$$

for some constant C. Finally choose $N = \min\{N_x, N_t\}$, and notice that the function $N^2 \exp(-\sqrt{\pi d\alpha N})$ is decreasing when $N > \frac{16}{\pi d\alpha N}$.

4.4. Fixed-point iteration

We now take up the existence proof of the solution of the discrete system by fixed-point iteration. The idea is to produce a sequence that converge to the solution of the K-S equation. By selecting an initial approximation U^0 we iterate the continuous map G repeatedly via the formula

$$U^{n+1} = G(U^n) + F^0, \quad n = 0, 1, 2, ... \tag{13}$$

Since $G(U^n) = -\Big(U^n \circ A_1U^n + \alpha A_2U^n + \beta A_4U^n\Big)B^T$, where $B = T_0\tilde{B}$ for some bounded matrix $\tilde{B}$. Choose T_0 sufficiently small such that $\| G(U) \| \leq k$ and $\| dG(U) \| < k$, for any U in any given fixed ball $\mathcal{B}$ about the origin, where k is a constant with $0 < k < 1$. Now

$$\begin{aligned} \| U^{n+1} - U^n \| &= \| G(U^n) - G(U^{n-1}) \| \\ &\leq k \| U^n - U^{n-1} \| \\ &\vdots \\ &\leq k^n \| U^1 - U^0 \| \end{aligned}$$

which implies that

$$\begin{aligned} \| U^{n+1} - U^0 \| &\leq k^n \| U^1 - U^0 \| + k^{n-1} \| U^1 - U^0 \| + ... + \| U^1 - U^0 \| \\ &\leq \frac{1}{1-k} \| U^1 - U^0 \| \end{aligned}$$

and for positive integer p, we have

$$\| U^{n+p} - U^n \| \leq \frac{k^p}{1-k} \| U^1 - U^0 \|$$

so, with a choice of $k \in (0, 1)$ and $U^0, U^1, ...$, we see that all iterates will remain in the ball

$$\mathcal{B} = \{V : \| V \| \leq \frac{1}{1-k} \| U^1 - U^0 \|\}.$$

Also there is an integer N such that $\| U^{n+p} - U^n \| < \epsilon$ for all $n > N$, and for any p. Therefore the sequence $\{U^n\}$ is a Cauchy sequence, and hence converges to some U^* where $\lim_{n\to\infty} U^n = U^*$. For uniqueness, suppose there are two distinct solutions, say U^* and $U^\star$, then using $\| dG(U) \| < k$, we have

$$\| U^\star - U^* \| = \| G(U^\star) - G(U^*) \| \leq k \| U^\star - U^* \|$$

for $k = 1/2$, we arrive at a contradiction, this shows that the solution is unique. With the notation as above we have proved the Theorem.

Theorem 4 *Given a constant $R > 0$, there is a constant $T_0 > 0$ such that if $\| U^1 - U^0 \| < R/2$, then the solution (12) has a unique solution. Moreover, the iteration scheme (13) with $U^0 = 0$ converges to this unique solution.*

4.5. Validation of Mesh-Free Sinc-collocation method

Consider the nonlinear K-S equation in (1), subject to the initial condition (2), and the boundary conditions

$$u(a,t)=0,\ u(b,t)=0,\ u_x(a,t)=0,\ u_x(b,x)=0,\ \ t>0. \tag{14}$$

To implement the Sinc-collocation method, following [20], we discretize time derivative of the nonlinear K-S equation using the Crank-Nicolson scheme, and space derivatives by the $\theta-$weighted $(\theta=1/2)$ scheme successive two time levels n and $n+1$

$$\Big(\frac{u^{n+1}-u^n}{\delta t}\Big)+\Big(\frac{(uu_x)^{n+1}+(uu_x)^n}{2}\Big)+\alpha\Big(\frac{(u_{xx})^{n+1}+(u_{xx})^n}{2}\Big)+\beta\Big(\frac{(u_{xxxx})^{n+1}+(u_{xxxx})^n}{2}\Big)=0 \tag{15}$$

where $u^n=u(x,t^n)$ is the value of the solution at the $n-$th time step, and $t^n=t^{n-1}+\delta t$, where δt is a time step size. The nonlinear term $(uu_x)^{n+1}$ must be linearized before continuing. This can be accomplished by using the following formula which obtained by applying the Taylor expansion, as follows

$$(u_x)^{n+1}\approx(u_x)^n+\delta t\frac{u_x^{n+1}-u_x^n}{\delta t}+\mathcal{O}(\delta t^2),\ (uu_x)^{n+1}\approx(uu_x)^n+\delta t\Big[(u_t)^nu_x^n+(u)^nu_{xt}^n\Big]+\mathcal{O}(\delta t^2)$$

which can be simplified to

$$(uu_x)^{n+1}\approx(uu_x)^n+\delta t\Big[u_x^n\frac{(u)^{n+1}-(u)^n}{\delta t}+u^n\frac{u_x^{n+1}-u_x^n}{\delta t}\Big]+\mathcal{O}(\delta t^2) \tag{16}$$

Finally, we arrive at the linearization

$$(uu_x)^{n+1}\approx(u)^{n+1}u_x^n+u_x^{n+1}u^n-u^nu_x^n \tag{17}$$

So equation (15) can be rewritten as

$$\Big(\frac{u^{n+1}-u^n}{\delta t}\Big)+\Big(\frac{u^nu_x^{n+1}+u^{n+1}u_x^n)}{2}\Big)+\alpha\Big(\frac{(u_{xx})^{n+1}+(u_{xx})^n}{2}\Big)+\beta\Big(\frac{(u_{xxxx})^{n+1}+(u_{xxxx})^n}{2}\Big)=0 \tag{18}$$

Rearranging equation (18), we get

$$u^{n+1}+\frac{\delta t}{2}\Big(u^nu_x^{n+1}+u^{n+1}u_x^n+\alpha(u_{xx})^{n+1}+\beta(u_{xxxx})^{n+1}\Big)=u^n-\frac{\delta t}{2}\Big(\alpha(u_{xx})^n+\beta(u_{xxxx})^n\Big) \tag{19}$$

where u^n are the $n-$th iterates of the approximate solutions. Now the space variable is discretized upon the use of Sinc-collocation at the points

$$\{x_1=a,...,x_i+a+(i-1)h,...,x_N=b\},\ \ h=\frac{|b-a|}{N-1} \tag{20}$$

The solution of equation (15) is interpolated and approximated by means of the Sinc functions as

$$u^n(x)=\sum_{j=0}^{N}u_j^nS_j(x),\ S_j(x)=\text{sinc}\left(\frac{x-(j-1)h-a}{h}\right) \tag{21}$$

The unknown parameters u_j in equation (21) are to be determined by collocation method. Therefore, for each collocation point x_i in (20), equation (21) can be written as

$$u^n(x_i)=\sum_{j=0}^{N}u_j^nS_j(x_i),\ i=1,...,N. \tag{22}$$

Substituting equation (22) into equation (18), for the interior points $x_i, i=1,...,N-1$, we get

$$\sum_{j=0}^{N} u_j^{n+1} S_j(x_i) + \tfrac{\delta t}{2}\Big[\sum_{j=0}^{N} u_j^n S_j(x_i) \sum_{j=0}^{N} u_j^{n+1} S_j'(x_i) +$$

$$\sum_{j=0}^{N} u_j^{n+1} S_j(x_i) \sum_{j=0}^{N} u_j^n S_j'(x_i) + \alpha \sum_{j=0}^{N} u_j^{n+1} S_j''(x_i) + \beta \sum_{j=0}^{N} u_j^{n+1} S_j''''(x_i)\Big] \tag{23}$$

$$= \sum_{j=0}^{N} u_j^n S_j(x_i) - \tfrac{\delta t}{2}\Big[\alpha \sum_{j=0}^{N} u_j^n S_j''(x_i) + \beta \sum_{j=0}^{N} u_j^n S_j''''(x_i)\Big]$$

The boundary conditions reads as

$$\sum_{j=0}^{N} u_j^{n+1} S_j(x_0) = \sum_{j=0}^{N} u_j^{n+1} S_j'(x_0) = 0, \;\; \sum_{j=0}^{N} u_j^{n+1} S_j(x_N) = \sum_{j=0}^{N} u_j^{n+1} S_j'(x_N) = 0. \tag{24}$$

The system (23) and (24) contain $N+1$ equations with $N+1$ unknowns u_j^n which can be easily solved by Gaussian elimination method. Once the values of u_j^n are obtained then the solution for u can be derived from equation (21). Now we switch to a matrix representation of Equations (23) and (24). Define the matrices

$$\{N_1^n\}_{ij} = \Big[\sum_{\ell=0}^{N} \delta_\ell^{(0)} u_\ell^n(x_i)\Big]\delta_j^{(1)}(x_i), \;\; \{N_2^n\}_{ij} = \Big[\sum_{\ell=0}^{N} \delta_\ell^{(1)} u_\ell^n(x_i)\Big]\delta_j^{(0)}(x_i)$$

The above two equations are valid for $i = 1, ..., N-2$, and in the rows $1, N-1$, and N the boundary conditions (24) hold true. So in matrix form equations (23), (24) can be written as

$$\Big[I^{(0)} + \frac{\delta t}{2}(N_1^n + N_2^n) + \frac{\delta t}{2}(\alpha I^{(2)} + \beta I^{(4)})\Big]u^{n+1} = \Big[I^{(0)} - \frac{\delta t}{2}(\alpha I^{(2)} + \beta I^{(4)})\Big]u^n + F^{n+1} \tag{25}$$

If the condition of the left hand side of equation (25) is small, then u^n for any time t_k can be easily calculated from the initial condition. For stability analysis of the Sinc-collocation solution, the evolution of error can be written as

$$\Big[I^{(0)} + \frac{\delta t}{2}(N_1^n + N_2^n) + \frac{\delta t}{2}(\alpha I^{(2)} + \beta I^{(4)})\Big]e^{n+1} = \Big[I^{(0)} - \frac{\delta t}{2}(\alpha I^{(2)} + \beta I^{(4)})\Big]e^n \tag{26}$$

where $e^n = |u_{exact}^n - u_{approx}^n|$, where u_{exact}^n and u_{approx}^n are the exact and Sinc-collocation approximated solutions at time t^k respectively. Equation (26) can be written as

$$e^{n+1} = M^{-1} N e^n \tag{27}$$

where

$$M^{-1}N = \Big[I^{(0)} + \frac{\delta t}{2}(N_1^n + N_2^n) + \frac{\delta t}{2}(\alpha I^{(2)} + \beta I^{(4)})\Big]^{-1}\Big[I^{(0)} - \frac{\delta t}{2}(\alpha I^{(2)} + \beta I^{(4)})\Big].$$

The scheme is considered numerically stable if $\rho(M^{-1}N) \leq 1$, where $\rho(.)$ denoted the spectral radius. Stability is assured if

$$\frac{1 - 0.5\delta t(\alpha\lambda_2 + \beta\lambda_4)}{1 + 0.5\delta t(\lambda_{N_1} + \lambda_{N_2}) + 0.5\delta t(\alpha\lambda_2 + \beta\lambda_4)} \leq 1 \tag{28}$$

where $\lambda_2, \lambda_4, \lambda_{N_1}, \lambda_{N_2}$ are the eigenvalues for the matrices $I^{(2)}, I^{(4)}, N_1^n, N_2^n$ respectively. In order to study stability of Sinc methods, we should find some bound for the eigenvalues of the matrices appeared in Equation (28). For a bound for the Toeplitz matrices $I^{(2)}$ and $I^{(4)}$, we refer the reader to [17] where well-known results for upper and lower bounds are established. While for other two matrices N_1 and N_2, the eigenvalues depends on the choice of the parameter N, that is already taken into account for the scheme. The stability of the scheme

and conditioning of the component matrices of the matrix $M^{-1}N$ depend on the weight parameter and the minimum distance between any two collocation points h in the domain set $[a, b]$.

Remark I: In the previous sections, we showed how to replace the derivatives and integrals by the necessary matrices if the boundary conditions are homogeneous. In particular, the Sinc methodology presented in this paper is still applicable to equation (1) with non-homogeneous boundary conditions. The non-homogeneous boundary conditions can be transformed to homogenous boundary conditions by the change of variables $w(x,t) = u(x,t) - \chi(x,t)$, where $\chi(x,t)$ is an analytic function that is defined as in Lemma 5.1 of [21].

5. RESULTS AND ANALYSIS

Choosing examples with known solutions allows for a more complete error analysis. In order to assess the advantages of the proposed methods, Sinc-Galerkin method over the mesh-free Sinc-collocation method in terms of accuracy and efficiency for solving K-S equation, we have applied the two methods to two different examples. For the numerical results:

Sinc-Galerkin method(SGM): We apply the Sinc-Galerkin method which has the matrix form (9), In all calculations, we have used $d = \frac{\pi}{2}, \alpha = \frac{1}{2}, N_x = N_t = 64$, and the step-sizes h_x, h_t can be determined by $h_x = \sqrt{\pi d/(\alpha N_x)}, h_t = \sqrt{\pi d/(\alpha N_t)}$. One advantages of the Sinc method is that it automatically determines the graded mesh.

Mesh-free Sinc-collocationn method(SCM): We also solve the K-S equation using the mesh-free Sinc-collocation method (25). In our computational work, we take time step sizes $\delta t = 0.05$ through the interval $[-5, 5]$ and $N = 100$ for the set of collocation points as in equation (20). The step-size h is the minimum distance between any two points in equation (20). The computations associated with the two examples were performed using Mathematica.

Example 1 *Consider the equation*

$$u_t + uu_x + 2u_{xx} + u_{xxxx} = 0 \tag{29}$$

Where we set $\alpha = 2$ and $\beta = 1$ into equation (1). This problem has exact solution [14]

$$u(x,t) = -\frac{1}{\kappa} + \frac{60}{19}\kappa(-38\kappa^2 + 2)\tanh\theta + 120\kappa^3\tanh^3\theta$$

where $\theta = \kappa x + t$ and $\kappa = \frac{1}{2}\sqrt{22/19}$. We will use this solution, evaluated at $t = 0$, as the initial condition, also we extract the required boundary conditions from the exact solution on the interval $[-10, 10]$.
From the numerical results in Table 1, it can be seen that the approximate solution (using either method) is quite close to the exact solution. This shows the approximate solution is efficiency. A surface plot of the numerical solution is shown in Figure 1 using Sinc-Galerkin method.

Table 1. Comparison results for Example 1

t	x	Exact	Sinc-Galerkin	Mesh-Free-Collocation
0.25	-6	-5.04757	-5.04758	-5.04758
	-4	-3.58451	-3.58448	-3.58449
	-2	2.68033	2.68962	2.68963
	0	-5.32919	-5.32005	-5.32010
	2	-2.90157	-2.90110	-2.90113
	4	0.89126	0.891233	0.891232
	6	1.46202	1.46202	1.46202
0.5	-6	-4.91406	-4.91410	-4.91408
	-4	-2.64236	-2.64271	-2.64246
	-2	3.46494	3.46144	3.46140
	0	-7.08063	-7.08060	-7.08061
	2	-1.4381	-1.43859	-1.43862
	4	1.14153	1.14138	1.14128
	6	1.49241	1.49252	1.49250

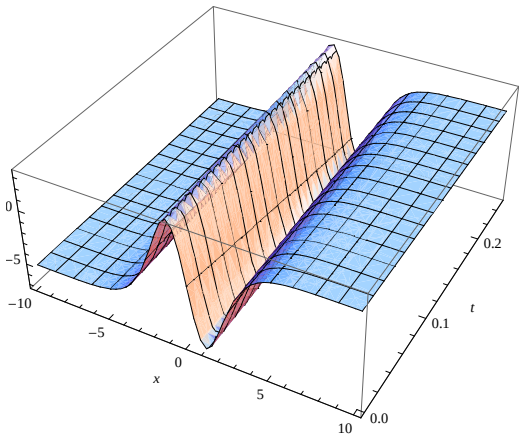


Figure 1. The Sinc-Galerkin solution on $-10 < x < 10,\ 0 < t < 0.25$, for Example 1

Remark II: We have shown in Theorem 1 that the problem has a local solution in the interval $(0, T)$ provided that T is sufficiently small. It might be possible that the scheme will diverge for large T. To run the scheme, we find a smaller time interval say $(0, T_1)$, in which the scheme will converge, and solving Equation (1) using the given initial condition (2). Then we find a $T_2 > 0$ and solve the system over the interval (T_1, T_2), where the initial condition now is the solution found in the interval $(0, T_1)$ evaluated at $t = T_1$. This means that the system so far has a solution in the interval $(0, T_2)$. Continuing in this way, we generate a sequence T_1, T_2, T_3, to get for (1) defined for all $0 < t < T$ such that $T_1 \leq T_2 \leq T_3 \leq ... \leq T$.

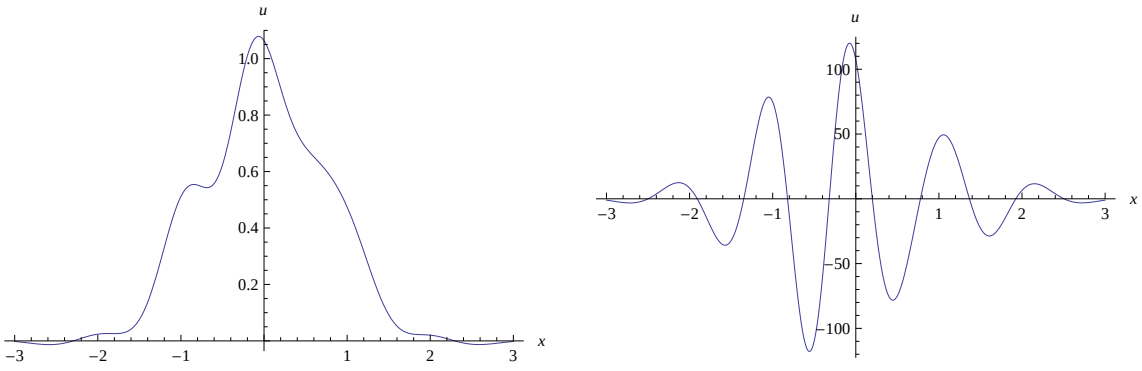
Example 2 *As a second example, we consider equation (1) with $\alpha = \beta = 1$, subject to the Gaussian initial condition*

$$u(x, 0) = \exp(-x^2) \tag{30}$$

with boundary conditions

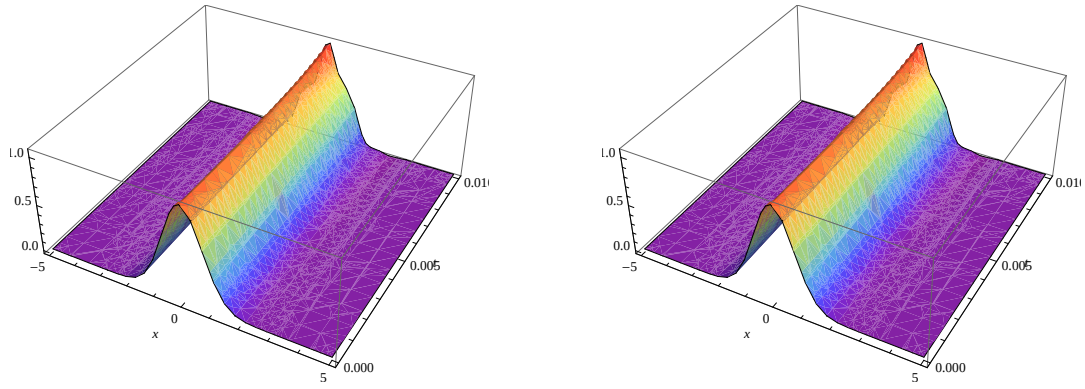
$$u(-5, t) = 0,\ u(5, t) = 0,\ u_x(5, t) = 0,\ u_x(5, x) = 0,\ \ t > 0.$$

The K-S equation subject to the Gaussian initial condition (30) exhibiting the chaotic behavior over a finite spatial domain. The numerical results are presented in Figures 2 and 3. A surface plot of the numerical solution is shown in Figure 4 using Sinc-Galerkin method, and Figure 5 using Mesh-free Sinc-collocation.



(a) (b)

Figure 2. (a) The chaotic solution with Gaussian initial condition at $t = 1$ for Example 2 by Sinc-Galerkin, (b) The chaotic solution with Gaussian initial condition at $t = 3$ for Example 2 by Mesh-Free, with $N = 160$



(a) (b)

Figure 3. (a) The solution $u(x,t)$ Sinc-Galerkin method $N_x = 64, N_t = 64, T_0 = 1$ for Example 2, (b) The solution $u(x,t)$ using Mesh-free Sinc-collocation method $N = 160$ for Example 2

6. CONCLUSION

The fundamental goal of this paper is to propose efficient algorithms for solving the K-S equation. Both the Sinc-Galerkin method and Mesh-free Sinc-collocation method were described in details, and implemented to compute a numerical solution to the K-S equation. A convergence proof was provided for the Sinc-Galerkin, and a brief stability analysis was provided for the mesh-free Sinc-collocation method, which produced a necessary condition for stability of the method. The efficiency of the methods was tested on two examples. The results obtained by both methods are very close to analytical ones. The algorithms were found to be stable, exponentially convergent in space and reliable numerical method for solving the K-S equation. The numerical results illustrate that the two methods are equivalent theoretically.

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