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Homogeneous components of a CDH fuzzy space

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ABSTRACT

We prove that fuzzy homogeneous components of a CDH fuzzy topological space $(X, \Im)$ are clopen and also they are CDH topological subspaces of its 0-cut topological space $(X, \Im_0)$.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we follow the notions and terminologies as they appeared in [1]. As defined in [2], the notion of a fuzzy set in a set X is a function from X into the closed interval [0,1]. Accordingly, Chang [3] introduced the notion of a fuzzy topological space on a non-empty set X as a collection of fuzzy sets on X, closed under arbitrary suprema and finite infima and containing the constant fuzzy sets 0 and 1. Mathematicians extended many topological concepts to include fuzzy topological spaces such as: separation axioms, connectedness, compactness and metrizability, see [4]-[9]. Several fuzzy homogeneity concepts were discussed in [1, 10-17].

A separable topological space (X, τ) is countable dense homogeneous (CDH) [18] if given any two countable dense subsets A and B of (X, τ) there is a homeomorphism $f: (X, \tau) \to (X, \tau)$ such that $f(A) = B$. Recently, authors in [1] extended CDH topological property to include fuzzy topological spaces. They proved that their extension is a good extension in the sense of Lowen, and proved that a-cut topological space $(X, \Im_a)$ of a CDH fuzzy topological space $(X, \Im)$ is CDH in general only for $a = 0$.

In the present work, we show that fuzzy homogeneous components of a CDH fuzzy topological space are clopen and also they are CDH in its 0-cut topological space $(X, \Im_0)$. Given a topological space (X, τ), the relation τ, defined as for $x, y \in X, x \, \tau \, y$ iff there exists a homeomorphism $h: (X, \tau) \to (X, \tau)$ such that $h(x) = y$, turns out to be an equivalence relation on X. The equivalence class of x under it is denoted as C_x^τ and is called the homogeneous component of (X,τ) at x. Analogously, we define the fuzzy homogeneous component $C_x^\Im$ of x, for a fuzzy topological space $(X, \Im)$ (with homeomorphism replaced by fuzzy homeomorphism). The following propositions will be used in the sequel:

a. Proposition 1.1.

Let $(X, \Im)$ be a fuzzy topological space. If $C_x^\Im$ is a fuzzy homogeneous component of $(X, \Im)$ and U is a non-empty open subset of $(X, \Im_a)$ with $U \subseteq C_x^\Im$, then $C_x^\Im \in \Im_a$ [11].

b. Proposition 1.2.

Let $(X,\mathfrak{I})$ be a fuzzy topological space. Let A be a subset of X and let P be a collection of fuzzy points of X. Then we have the following [1]:

i. If A is dense in $(X,\mathfrak{I}_0)$, then $\mathbb{Q}(A)$ is dense(I) in $(X,\mathfrak{I})$.

ii. If P is dense(I) in $(X,\mathfrak{I})$, then $S(P)$ is dense in $(X,\mathfrak{I}_0)$.

c. Proposition 1.3.

If $(X,\mathfrak{I})$ is a fuzzy topological space and $C_x^{\mathfrak{I}}$ is a fuzzy homogeneous component of $(X,\mathfrak{I})$, then for any fuzzy homeomorphism $h:(X,\mathfrak{I})\to(X,\mathfrak{I})$, $h(C_x^{\mathfrak{I}})=C_x^{\mathfrak{I}}$ [11].

d. Proposition 1.4.

If $(X,\mathfrak{I})$ is a CDH fuzzy topological space, then $(X,\mathfrak{I}_0)$ is CDH [1].

e. Proposition 1.5.

Let $(X,\mathfrak{I})$ be a fuzzy topological space and let $f:(X,\mathfrak{I})\to(X,\mathfrak{I})$ be a fuzzy continuous (homeomorphism) map. Then $f:(X,\mathfrak{I}_a)\to(X,\mathfrak{I}_a)$ is continuous (homeomorphism) for all $a\in[0,1)$ [19].

2. RESULTS

a. Theorem 2.1.

Let (X,τ) be a CDH topological space. Then every homogeneous component C_x^{τ} is clopen. In this paper, we will mainly obtain a fuzzy version of Theorem 2.1. This fuzzy version is as follows [20]:

b. Theorem 2.2.

If $(X,\mathfrak{I})$ is a CDH fuzzy topological space, then every fuzzy homogeneous component $C_x^{\mathfrak{I}}$ of $(X,\mathfrak{I})$ is clopen in $(X,\mathfrak{I}_0)$. For any fuzzy topological space $(X,\mathfrak{I})$ and any $x\in X$, author in [11] proved that $C_x^{\mathfrak{I}}\subseteq C_x^{\mathfrak{I}_0}$. If for all $x\in X$, $C_x^{\mathfrak{I}}=C_x^{\mathfrak{I}_0}$, then Theorem 2.2 follows obviously using Theorem 2.1. Therefore, the following question is important:

c. Question 2.3. Let $(X,\mathfrak{I})$ be a CDH fuzzy topological space. Is it true that $C_x^{\mathfrak{I}}=C_x^{\mathfrak{I}_0}$ for all $x\in X$.

The following example gives a negative answer of Question 2.3:

d. Example 2.4. For fixed $0<a<1$, let $X=\{x,y\}$ and define $\mathfrak{I}=\{0,1,x_{\frac{a}{2}},y_{\frac{a}{4}},x_{\frac{a}{2}}\cup y_{\frac{a}{4}}\}$. Then $(X,\mathfrak{I})$ is CDH and $C_x^{\mathfrak{I}}=\{x\}$ but $C_x^{\mathfrak{I}_0}=X$ for all $x\in X$.

The following two lemmas will be used in the following main result:

e. Lemma 2.5. Let $(X,\mathfrak{I})$ be a fuzzy topological space and let $C_x^{\mathfrak{I}}$ be a fuzzy homogeneous component of $(X,\mathfrak{I})$ with $C_x^{\mathfrak{I}}\notin\mathfrak{I}_0$. Then $X-C_x^{\mathfrak{I}}$ is dense in $(X,\mathfrak{I}_0)$.

Proof. Assume on the contrary that $X-C_x^{\mathfrak{I}}$ is not dense in $(X,\mathfrak{I}_0)$. Then there exists a non-empty set $U\in\mathfrak{I}_0$ such that $U\cap(X-C_x^{\mathfrak{I}})=\emptyset$. Thus $U\subseteq C_x^{\mathfrak{I}}$ and by Proposition 1.1 we have $C_x^{\mathfrak{I}}\in\mathfrak{I}_0$, a contradiction.

f. Lemma 2.6. Let $(X,\mathfrak{I})$ be a CDH fuzzy topological space. Suppose that there exists a fuzzy homogeneous component $C_x^{\mathfrak{I}}$ of $(X,\mathfrak{I})$ with $C_x^{\mathfrak{I}}\notin\mathfrak{I}_0$. Let S be a countable dense subset of $(X,\mathfrak{I}_0)$. Let

$$\mathrm{D}=\mathrm{X}-Cl(\mathrm{S}\cap(X-C_x^{\mathfrak{I}}))$$

and

$$T=((D\cap S)\cup(S\cap(X-C_x^{\mathfrak{I}})))-Bd(D)$$

where the closure and the boundary are taking in $(X,\mathfrak{I}_0)$. Then

i. $D\cap S\subseteq C_x^{\mathfrak{I}}$

ii. $\left(X-Cl(D)\right)\cap T\subseteq X-C_x^{\mathfrak{I}}$,

iii. $D\neq\emptyset$,

iv. T is dense in $(X,\mathfrak{I}_0)$.

Proof. i) Since $S\cap(X-C_x^{\mathfrak{I}})\subseteq Cl(S\cap(X-C_x^{\mathfrak{I}})$, then $D\subseteq X-(S\cap(X-C_x^{\mathfrak{I}}))\subseteq(X-S)\cup C_x^{\mathfrak{I}}$. Thus $D\cap S\subseteq\emptyset\cup\left(C_x^{\mathfrak{I}}\cap S\right)\subseteq C_x^{\mathfrak{I}}$.

ii) Let $t\in(X-Cl(D))\cap T$. Since $t\in X-Cl(D)$, then $t\notin D$. Since $t\in T$, then $t\in S\cap\left(X-C_x^{\mathfrak{I}}\right)\subseteq X-C_x^{\mathfrak{I}}$.

iii) Suppose on the contrary that $D=\emptyset$. Then $\mathrm{Cl}(\mathrm{S}\cap(X-C_x^{\mathfrak{I}}))=X$. Let $A=S\cap(X-C_x^{\mathfrak{I}})$ and $B=A\cup\{x\}$. Then A,B are both dense in $(X,\mathfrak{I}_0)$ and by Proposition 1.2 (i), $\mathbb{Q}(A)$ and $\mathbb{Q}(B)$ are both dense(I) in $(X,\mathfrak{I})$. Since $(X,\mathfrak{I})$ is CDH, then there is a fuzzy homeomorphism $h:(X,\mathfrak{I})\to(X,\mathfrak{I})$

such that $h(S(\mathbb{Q}(B))) = S(\mathbb{Q}(A))$. So $h(B) = A$. Since $x \in C_x^{\mathfrak{I}}$, then $h(x) \in h(C_x^{\mathfrak{I}})$. By Proposition 1.3, $h(C_x^{\mathfrak{I}}) = C_x^{\mathfrak{I}}$ and so $h(x) \in C_x^{\mathfrak{I}}$. On the other hand, $h(x) \in A$, that is $h(x) \notin C_x^{\mathfrak{I}}$, a contradiction.

iv) It is sufficient to show that $Cl(D) \subseteq Cl(T)$ and $X - Cl(D) \subseteq Cl(T)$.

To see that $Cl(D) \subseteq Cl(T)$, let $a \in Cl(D)$ and $U \in \mathfrak{I}_0$ with $a \in U$. Then $U \cap D \neq \emptyset$. Since S is dense in $(X, \mathfrak{I}_0)$ and $D \in \mathfrak{I}_0$, we have $U \cap (D \cap S) \neq \emptyset$. Since $D \in \mathfrak{I}_0$, then $D \cap Bd(D) = \emptyset$. Choose $b \in U \cap (D \cap S)$. If $b \notin T$, then $b \in D \cap Bd(D) = \emptyset$ and thus $b \in U \cap T$. Therefore, $a \in Cl(T)$.

To see that $X - Cl(D) \subseteq Cl(T)$, let $a \in X - Cl(D)$ and $U \in \mathfrak{I}_0$ with $a \in U$. Since $a \in X - Cl(D)$, then there is $V \in \mathfrak{I}_0$ such that $a \in V$ and $V \cap D = \emptyset$. On the other hand, $a \notin Cl(D)$ implies $a \notin D$ and hence $a \in Cl(S \cap (X - C_x^{\mathfrak{I}}))$. Thus, we have

$$(U \cap V) \cap (S \cap (X - C_x^{\mathfrak{I}})) \neq \emptyset.$$

Choose $b \in (U \cap V) \cap (S \cap (X - C_x^{\mathfrak{I}}))$. Since $b \in V$ and $V \cap D = \emptyset$, then $b \notin Bd(D)$ and hence $b \in U \cap T$. Therefore, $a \in Cl(T)$.

The following is the main result of this paper:

g. Theorem 2.7. If $(X, \mathfrak{I})$ is a CDH fuzzy topological space, then every fuzzy homogeneous component $C_x^{\mathfrak{I}}$ of $(X, \mathfrak{I})$ is open in $(X,\mathfrak{I}_0)$.

Proof. Suppose on the contrary that for some $x \in X, C_x^{\mathfrak{I}} \notin \mathfrak{I}_0$. Since $(X, \mathfrak{I})$ is CDH, then by Proposition 1.4, $(X, \mathfrak{I}_0)$ is CDH. Choose a countable dense set S of $(X, \mathfrak{I}_0)$. Let

$$D = X - Cl(S \cap (X - C_x^{\mathfrak{I}}))$$

and

$$T = ((D \cap S) \cup (S \cap (X - C_x^{\mathfrak{I}}))) - Bd(D).$$

By Lemma 2.5, $C_x^{\mathfrak{I}}$ is dense in $(X, \mathfrak{I}_0)$. Since $D \in \mathfrak{I}_0$ and by Lemma 2.6 (iii) $D \neq \emptyset$, then $D \cap (X - C_x^{\mathfrak{I}}) \neq \emptyset$. Choose $y \in D \cap (X - C_x^{\mathfrak{I}})$ and let

$$M = T \cup \{y\}.$$

Since $T \subseteq S$, then T and M are both countable. Also, by Lemma 2.6 (iv) T and M are dense in $(X, \mathfrak{I}_0)$. Then by Proposition 1.2 (i), $\mathbb{Q}(T)$ and $\mathbb{Q}(M)$ are two countable dense of the CDH fuzzy topological space $(X, \mathfrak{I})$. Thus there is a fuzzy homeomorphism $h: (X, \mathfrak{I}) \to (X, \mathfrak{I})$ such that $h(S(\mathbb{Q}(M))) = S(\mathbb{Q}(T))$. So $h(M) = T$. Since $y \in X - C_x^{\mathfrak{I}}$, then by Proposition 1.3, $h(y) \in X - C_x^{\mathfrak{I}}$. Since $h(y) \in T \subseteq S$, then $h(y) \in S \cap (X - C_x^{\mathfrak{I}}) \subseteq Cl(S \cap (X - C_x^{\mathfrak{I}}))$ and so $h(y) \in X - D = Cl(X - D)$. Since $h(y) \in T$, then $h(y) \in X - Bd(D)$. Therefore, $h(y) \in X - Cl(D)$. Set $O = h^{-1}(X - Cl(D))$. By Proposition 1.5, $h: (X, \mathfrak{I}_0) \to (X, \mathfrak{I}_0)$ is continuous at y and so there exists $O \in \mathfrak{I}_0$ such that $y \in O$ and $h(O) \subseteq X - Cl(D)$. Since $y \in D \cap O \in \mathfrak{I}_0$ and T is dense in $(X, \mathfrak{I}_0)$, then there exists $z \in D \cap O \cap T \subseteq D \cap S$. By Lemma 2.6 (i), we have $z \in C_x^{\mathfrak{I}}$ and by Proposition 1.3 $h(z) \in C_x^{\mathfrak{I}}$. Since $z \in O$, $h(z) \in h(O) \subseteq X - Cl(D)$. Also, since $z \in T \subseteq M$, $h(z) \in h(M) = T$. Therefore, $h(z) \in (X - Cl(D)) \cap T$ and by Lemma 2.6 (ii), $h(z) \in X - C_x^{\mathfrak{I}}$, a contradiction.

h. Corollary 2.8. If $(X, \mathfrak{I})$ is a CDH fuzzy topological space, then every fuzzy homogeneous component $C_x^{\mathfrak{I}}$ of $(X, \mathfrak{I})$ is clopen in $(X,\mathfrak{I}_0)$.

Recall that a fuzzy topological space $(X, \mathfrak{I})$ is said to be homogeneous [16] if for any two points x_1, x_2 in X, there exists a fuzzy homeomorphism $h: (X, \mathfrak{I}) \to (X, \mathfrak{I})$ such that $h(x_1) = x_2$. A fuzzy topological space $(X, \mathfrak{I})$ is homogeneous iff $C_x^{\mathfrak{I}} = X$ for all $x \in X$.

i. Corollary 2.9. If $(X, \mathfrak{I})$ is a CDH fuzzy topological space and $(X, \mathfrak{I}_0)$ is connected, then $(X, \mathfrak{I})$ is homogeneous.

Proof. Let $x \in X$. According to Corollary 2.8, $C_x^{\mathfrak{I}}$ is clopen in $(X, \mathfrak{I}_0)$, and since $(X, \mathfrak{I}_0)$ is connected, then $C_x^{\mathfrak{I}} = X$.

j. Lemma 2.10. Let (X, τ) be a topological space and let C be a non-empty clopen subset of X. If D is a dense subset of (X, τ) and A is a dense subset of the subspace (C, τ_C), then $A \cup (D - C)$ is dense in (X, τ).

Proof. Suppose on the contrary that there is $U \in \tau - \{\emptyset\}$ such that $U \cap (D - C) = U \cap (X - C) \cap D = \emptyset$ and $U \cap A = \emptyset$. Since $U \cap (X - C) \in \tau$ and D is a dense in (X, τ), then $U \cap (X - C) = \emptyset$ and $U \subseteq C$. It follows that $U \in \tau_C - \{\emptyset\}$. Since A is a dense subset of (C, τ_C), then $U \cap A \neq \emptyset$, a contradiction.

k. Theorem 2.11. If $(X,\Im)$ is a CDH fuzzy topological space and $C_x^{\Im}$ is a fuzzy homogeneous component of $(X,\Im)$, then $(C_x^{\Im},(\Im_0)_{C_x^{\Im}})$ is a CDH topological space.

Proof. According to Corollary 2.8, $C_x^{\Im}$ is clopen in $(X,\Im_0)$ and hence $(C_x^{\Im},(\Im_0)_{C_x^{\Im}})$ is separable. Let A and B be any two countable dense subsets of $(C_x^{\Im},(\Im_0)_{C_x^{\Im}})$ and let P be a countable dense(I) collection of fuzzy points of $(X,\Im)$. Let $A_1 = A \cup (S(P) - C_x^{\Im})$ and $B_1 = B \cup (S(P) - C_x^{\Im})$. By Proposition 1.2 (ii) $S(P)$ is dense in $(X,\Im_0)$. Thus by Lemma 2.10, A_1 and B_1 are dense subsets of $(X,\Im_0)$. By Proposition 1.2 (i) $\mathbb{Q}(A_1)$ and $\mathbb{Q}(B_1)$ are dense(I) in $(X,\Im)$. Since $\mathbb{Q}(A_1)$ and $\mathbb{Q}(B_1)$ are clearly countable, there is a fuzzy homeomorphism $h:(X,\Im)\rightarrow(X,\Im)$ such that $h(A_1)=B_1$. Applying Proposition 1.3 to conclude that $h(A)=B$. By Proposition 1.5, $h:(X,\Im_0)\rightarrow(X,\Im_0)$ is a homeomorphism. Define $g:(C_x^{\Im},(\Im_0)_{C_x^{\Im}})\rightarrow(C_x^{\Im},(\Im_0)_{C_x^{\Im}})$ to be the restriction of h on $C_x^{\Im}$. Then g is a homeomorphism with $g(A)=h(A)=B$.

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