International Journal of Electrical and Computer Engineering (IJECE)
Vol. 9, No. 4, August 2019, pp. 3256~3261
ISSN: 2088-8708, DOI: 10.11591/ijece.v9i4.pp3256-3261

Densely homogeneous fuzzy spaces

Samer Al Ghour
Department of Mathematics and Statistics, Jordan University of Science and Technology, Jordan

Article Info

Article history:

Received Jul 26, 2018
Revised Apr 2, 2019
Accepted Apr 11, 2019

Keywords:

Cut topologies
Densely homogeneous
Fuzzy CDH
Good extension

ABSTRACT

We extend the concept of being densely homogeneous to include fuzzy topological spaces. We prove that our extension is a good extension in the sense of Lowen. We prove that a-cut topological space $(X, \Im_a)$ of a DH fuzzy topological space $(X, \Im)$ is DH in general only for $a = 0$.

Corresponding Author:

Samer Al Ghour,
Department of Mathematics and Statistics,
Jordan University of Science and Technology,
Irbid 22110, Jordan
Email: algore@just.edu.jo

1. INTRODUCTION

As defined in [1], the notion of a fuzzy set in a set X is a function from X into the closed interval [0,1]. Accordingly, Chang [2] introduced the notion of a fuzzy topological space on a non-empty set X as a collection of fuzzy sets on X, closed under arbitrary suprema and finite infima and containing the constant fuzzy sets 0 and 1. Mathematicians extended many topological concepts to include fuzzy topological spaces such as: separation axioms, connectedness, compactness and metrizability. Several fuzzy homogeneity concepts were discussed in [3-11]. A separable topological space (X, τ) is countable dense homogeneous (CDH) [12] if given any two countable dense subsets A and B of (X, τ) there is a homeomorphism $f: (X, \tau) \to (X, \tau)$ such that $f(A) = B$.

The study of CDH topological spaces and their related concepts is still a hot area of research, as appears in [13-20] and other papers. Recently, authors in [9] extended CDH topological property to include fuzzy topological spaces. They proved that their extension is a good extension in the sense of Lowen, and proved that a-cut topological space $(X, \Im_a)$ of a CDH fuzzy topological space $(X, \Im)$ is CDH in general only for $a = 0$. For the purpose of dealing with non-separable topological spaces, authors in [21] modified the definition of CDH topological spaces as follows: A subset A of a topological space (X, τ) is called a σ-discrete set if it is the union of countably many sets, each with the relative topology, being a discrete topological space. A topological space (X, τ) is densely homogeneous (DH) provided (X, τ) has a σ-discrete subset which is dense in (X, τ) and if A and B are two such σ-discrete subsets of (X, τ) there is a homeomorphism $f: (X, \tau) \to (X, \tau)$ such that $f(A) = B$. It is known that CDH and DH topological concepts are independent. The study of DH topological spaces is continued in [22-28] and other papers. As a main goal of the present work we will show how the definition of DH topological spaces can be modified in order to define a good extension of it in fuzzy topological spaces. We will give relationships between CDH and DH fuzzy.

Throughout this paper, if X is a set, then $|X| = Card\, X$ will denote its cardinality. We write $\mathbb{Q}$ (resp. $\mathbb{N}$) to denote the set of all rational numbers (resp. natural numbers). The closure of a fuzzy set λ in a fuzzy topological space $(X, \Im)$ will be denoted by $Cl(\lambda)$. Associated with a given topological space (X, τ) and arbitrary subset A of X, we denote the relative topology on A by τ_A, the closure of A by $Cl(A)$ and the boundary of A by $Bd(A)$.topological spaces as well as we will deal with cut topological spaces.

2. PRELIMINARIES

In this paper we shall follow the notations and definitions of [29] and [30]. If (X, τ) is a topological space, then the class of all lower semi-continuous functions from (X, τ) to $([0,1], \tau_u)$, where τ_u is the usual Euclidean topology on [0,1], is a fuzzy topology on X. This fuzzy topology is denoted by $\omega(\tau)$. The following definitions and propositions will be used in the sequel:

Definition 2.1. [9] Let X be a non-empty set, A be a non-empty subset of X and P be a collection of fuzzy points in X. Then

- $\mathbb{Q}(A)$ will denote the set
  $\mathbb{Q}(A) = \{x_r : x_r$ is a fuzzy point with $x \in A$ and $r \in \mathbb{Q} \cap (0,1)\}$.
- The support of P, denoted by $S(P)$, is defined by
  $S(P) = \{x : x_a \in P$ for some $a\}$.

Definition 2.2. [21] A subset A of a topological space (X, τ) is called a σ-discrete set if it is the union of countably many sets, each with the relative topology, being a discrete topological space. Definition 2.3. [21] A topological space (X, τ) is called densely homogeneous (DH) iff

- X has a σ-discrete dense subset.
- If A and B are two σ-discrete dense subsets of X, then there is a homeomorphism $h: (X, \tau) \rightarrow (X, \tau)$ such that $h(A) = B$.

Definition 2.4. [31] Associated with a given fuzzy topological space $(X, \Im)$ and arbitrary subset M of X, we define the induced fuzzy topology on M or the relative topology on M by

$$\Im_M = \{\lambda \mid M : \lambda \in \Im\}.$$

Definition 2.5. [9] A fuzzy topological space $(X, \Im)$ is said to be semi-discrete iff for any $x \in X$, there exists a fuzzy point or a fuzzy crisp point x_a for some a with $x_a \in \Im$. Definition 2.6. [32] Let $(X, \Im)$ be a fuzzy topological space and let P be a collection of fuzzy points of X. Then P is said to be

- Dense(I) if for every non-zero fuzzy open set λ there exists $p \in P$ such that $p \in \lambda$.
- Dense(II) if $Cl(\bigcup_{p \in P} p) = 1$.

Definition 2.7. [9] A fuzzy topological space $(X, \Im)$ is called separable iff there exists a countable dense(I) collection of fuzzy points of X. Definition 2.8. [33] A property $\mathcal{P}_f$ of a fuzzy topological space is said to be a good extension of the property $\mathcal{P}$ in classical topology iff whenever the fuzzy topological space is topologically generated, say by (X, τ), then $(X, \omega(\tau))$ has property $\mathcal{P}_f$ iff (X, τ) has property $\mathcal{P}$.

Definition 2.9. [34] Let $(X, \Im)$ be a fuzzy topological space and $a \in [0,1)$. The topology $\{\lambda^{-1}(a, 1] : \lambda \in \Im\}$ on X is called a-cut topological space of $(X, \Im)$ and will be denoted by $\Im_a$. The topological space $(X, \Im_a)$ will be called a-cut topological space of $(X, \Im)$. Definition 2.10. [9] A fuzzy topological space $(X, \Im)$ is said to be countable dense homogeneous; denoted CDH; iff

- $(X, \Im)$ is separable.
- If P and W are two countable dense(I) collections of fuzzy points of X, then there is a fuzzy homeomorphism $h: (X, \Im) \rightarrow (X, \Im)$ such that $h(S(P)) = S(W)$.

Proposition 2.11. [9] Let $(X, \Im)$ be a fuzzy topological space and let P be a collection of fuzzy points of X. Then we have the following

- If P is dense (I), then $\mathbb{Q}(S(P))$ is dense(II).
- If P is dense (II), then $\mathbb{Q}(S(P))$ is dense(I).

Proposition 2.12. [9] Let (X, τ) be a topological space, $A \subseteq X$, and P be a collection of fuzzy points of X. Then we have the following

- If A is dense in (X, τ), then $\mathbb{Q}(A)$ is dense(I) in $(X, \omega(\tau))$.
- If P is dense(I) in $(X, \omega(\tau))$, then $S(P)$ is dense in (X, τ).

Proposition 2.13. [35] Let (X, τ_1) and (Y, τ_2) be two topological spaces. Then $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ is continuous iff $f: (X, \omega(\tau_1)) \rightarrow (Y, \omega(\tau_2))$ is fuzzy continuous. Proposition 2.14. [9] Let $f: (X, \Im_1) \rightarrow (Y, \Im_2)$ be a fuzzy homeomorphism map and P be a collection of fuzzy points of X. Then we have the following

– $S(f(P)) = f(S(P))$.
– If P is dense (I) in $(X,\Im_1)$, then $f(P)$ is dense(I) in $(Y,\Im_2)$.

Proposition 2.15. [9] Let $(X,\Im)$ be a semi-discrete fuzzy topological space. Then we have the following
– If P is countable dense (I) in $(X,\Im)$, then $S(P) = X$.
– $(X,\Im)$ is separable iff X is countable.

Proposition 2.16. [9] Let X be a countable set and let $(X,\Im)$ be a fuzzy topological space. Then $(X,\Im)$ is CDH iff $(X,\Im)$ is a semi-discrete fuzzy topological space. Proposition 2.17. [34] Let $(X,\Im)$ be a fuzzy topological space and let $f:(X,\Im) \to (X,\Im)$ be a fuzzy continuous (homeomorphism) map. Then $f:(X,\Im_a) \to (X,\Im_a)$ is continuous (homeomorphism) for all $a \in [0,1)$. Proposition 2.18. [9] Let $(X,\Im)$ be a fuzzy topological space. Let A be a subset of X and let P be a collection of fuzzy points of X. Then we have the following
– If A is dense in $(X,\Im_0)$, then $\mathbb{Q}(A)$ is dense(I) in $(X,\Im)$.
– If P is dense(I) in $(X,\Im)$, then $S(P)$ is dense in $(X,\Im_0)$.

3. DH FUZZY TOPOLOGICAL SPACES

In this section, we will define DH fuzzy topological spaces. We will prove that our new concept is a fuzzy topological property and a good extension of DH topological property in the sense of Lowen. Definition 3.1. A collection P of fuzzy points of a fuzzy topological space $(X,\Im)$ is said to be
– σ-semi-discrete iff $S(P) = \bigcup_{n=1}^{\infty} A_n$ with $(A_n, \Im_{A_n})$ is semi-discrete for all $n \in \mathbb{N}$.
– σ-semi-discrete dense (I) iff P is σ-semi-discrete and P is dense (I).
– σ-semi-discrete dense (II) iff P is σ-semi-discrete and P is dense (II).

Definition 3.2. A fuzzy topological space $(X,\Im)$ is said to be densely homogeneous (DH) iff
– $(X,\Im)$ has a σ-semi-discrete dense(I) collection of fuzzy points.
– If P and W are two σ-semi-discrete dense (I) collections of fuzzy points of $(X,\Im)$, then there is a fuzzy homeomorphism $h:(X,\Im) \to (X,\Im)$ such that $h(S(P)) = S(W)$.

Lemma 3.3. Let $(X,\Im)$ be a fuzzy topological space and P be a σ-semi-discrete collection of fuzzy points of X. Then $\mathbb{Q}(S(P))$ is a σ-semi-discrete collection of fuzzy points of $(X,\Im)$. Proof. It is easy to see that $S(P) = S(\mathbb{Q}(S(P)))$ and hence the result is obvious. Theorem 3.4. A fuzzy topological space $(X,\Im)$ is DH iff
– $(X,\Im)$ has a σ-semi-discrete dense(II) collection of fuzzy points.
– If P and W are two σ-semi-discrete dense (II) collections of fuzzy points of $(X,\Im)$, then there is a fuzzy homeomorphism $h:(X,\Im) \to (X,\Im)$ such that $h(S(P)) = S(W)$.

Proof. If $(X,\Im)$ is DH, then $(X,\Im)$ has a σ-semi-discrete dense (I) collection of fuzzy points P. By Proposition 2.11 (i), $\mathbb{Q}(S(P))$ is dense (II) and by Lemma 3.3, $\mathbb{Q}(S(P))$ is σ-semi-discrete. Let P and W be any two σ-semi-discrete dense (II) collections of fuzzy points of $(X,\Im)$. Then by Proposition 2.11 (ii) and Lemma 3.3, $\mathbb{Q}(S(P))$ and $\mathbb{Q}(S(W))$ are both σ-semi-discrete dense (I) collections of fuzzy points of $(X,\Im)$. Then there is a fuzzy homeomorphism $h:(X,\Im) \to (X,\Im)$ such that $h(S(\mathbb{Q}(S(P)))) = S(\mathbb{Q}(S(W)))$. Thus, $h(S(P)) = S(W)$. The proof of the other direction of this theorem is similar to the above one. Lemma 3.5. Let (X,τ) be a topological space. Let A be a non-empty subset of X and P be a collection of fuzzy points of X. Then
– τ_A is the discrete topology iff $(A,\omega(\tau)_A)$ is semi-discrete.
– If A is σ-discrete in (X,τ), then $\mathbb{Q}(A)$ is σ-semi-discrete in $(X,\omega(\tau))$.
– If P is σ-semi-discrete in $(X,\omega(\tau))$, then $S(P)$ is σ-discrete in (X,τ).

Proof. (i) Suppose that τ_A is the discrete topology and let $x \in A$. Then there exists $U \in \tau$ such that $\{x\} = U \cap A$. So, $\mathcal{X}_U \cap \mathcal{X}_A = \mathcal{X}_{U\cap A} = \mathcal{X}_{\{x\}} \in \omega(\tau)_A$. But clearly $\omega(\tau)_A$ is the crisp point with support x. Conversely, suppose that $(A,\omega(\tau)_A)$ is a semi-discrete fuzzy topological space and let $x \in A$. Then there exists a fuzzy point or a fuzzy crisp point x_a such that $x_a \in \omega(\tau)_A$. Choose $\lambda \in \omega(\tau)$ such that $x_a = \lambda \cap \mathcal{X}_A$. Thus, $\{x\} = \lambda^{-1}(0,1] \cap A$ and hence $\{x\} \in \tau_A$.

(ii) Since A is σ-discrete in (X,τ), then $A = \bigcup_{n=1}^{\infty} A_n$ with τ_{A_n} is the discrete topology for all $n \in \mathbb{N}$. So, by part (i) $(A_n,\omega(\tau)_{A_n})$ is semi-discrete for all n. Since $S(\mathbb{Q}(A)) = A$, then $\mathbb{Q}(A)$ is σ-semi-discrete in $(X,\omega(\tau))$. (iii) Since P is σ-semi-discrete in $(X,\omega(\tau))$, then $S(P) = \bigcup_{n=1}^{\infty} A_n$ with $(A_n,\omega(\tau)_{A_n})$ is semi-discrete for all $n \in \mathbb{N}$. So, by part (i) τ_{A_n} is the discrete topology for all n. Therefore, $S(P)$ is σ-discrete in (X,τ).

Theorem 3.6. Let (X,τ) be a topological space. Then (X,τ) is DH iff $(X,\omega(\tau))$ is DH. Proof. Suppose that (X,τ) is DH. Then (X,τ) has a σ-discrete dense subset A. By Lemma 3.5 (ii) and

Proposition 2.12 (i), $\mathbb{Q}(A)$ is σ-semi-discrete dense (I) in $(X,\omega(\tau))$. Let P and W be two σ-semi-discrete dense(I) collections of fuzzy points of $(X,\omega(\tau))$. Then by Lemma 3.5 (iii) and Proposition 2.12 (ii), $S(P)$ and $S(W)$ are both σ-discrete dense subsets of (X,τ). Thus, there is a homeomorphism $h:(X,\tau)\to(X,\tau)$ such that $h(S(P))=S(W)$. Proposition 2.13 ends the proof of this direction.

Conversely if $(X,\omega(\tau))$ is DH, then $(X,\omega(\tau))$ has a σ-discrete dense (I) collection of fuzzy points P. By Lemma 3.5 (iii) and Proposition 2.12 (ii), $S(P)$ is σ-discrete dense in (X,τ). Let A and B be two σ-discrete dense subsets of (X,τ). Then by Lemma 3.5 (ii) and Proposition 2.12 (i), $\mathbb{Q}(A)$ and $\mathbb{Q}(B)$ are both σ-semi-discrete dense (I) collections of fuzzy points of $(X,\omega(\tau))$. Thus, there is a fuzzy homeomorphism $h:(X,\omega(\tau))\to(X,\omega(\tau))$ such that $h(S(\mathbb{Q}(A)))=S(\mathbb{Q}(A))$. So, $h(A)=B$. Proposition 2.13 ends the proof of this direction.

Corollary 3.7. DH in fuzzy topological spaces is a good extension of DH in topological spaces. Recall that a property $\mathcal{P}_f$ of fuzzy topological spaces is called a fuzzy topological property if whenever $(X,\Im_1)$ possesses $\mathcal{P}_f$ and $h:(X,\Im_1)\to(Y,\Im_2)$ is a fuzzy homeomorphism, then $(Y,\Im_2)$ possesses $\mathcal{P}_f$. Lemma 3.8. Let $f:X\to Y$ be a bijective map. Then

- For any two fuzzy sets λ,μ in X, $f(\lambda\cap\mu)=f(\lambda)\cap f(\mu)$.
- For any $A\subseteq X$, $f(\mathcal{X}_A)=\mathcal{X}_{f(A)}$.

Proof. Straightforward. Lemma 3.9. Let $f:(X,\Im_1)\to(Y,\Im_2)$ be a fuzzy homeomorphism. Let A be a non-empty subset of X and P be a collection of fuzzy points of X. Then

- If $(A,(\Im_1)_A)$ is semi-discrete, then $(f(A),(\Im_2)_{f(A)})$ is semi-discrete.
- If P is σ-semi-discrete, then $f(P)$ is σ-semi-discrete.

Proof. (i) Let $y\in f(A)$, say $y=f(x)$ for some $x\in A$. Since $(A,(\Im_1)_A)$ is semi-discrete, there exists $r\in(0,1]$ such that $x_r\in(\Im_1)_A$. Choose $\lambda\in\Im_1$ such that $x_r=\lambda\cap\mathcal{X}_A$. Then by Lemma 3.8 $y_r=(f(x))_r=f(x_r)=f(\lambda\cap\mathcal{X}_A)=f(\lambda)\cap f(\mathcal{X}_A)=f(\lambda)\cap\mathcal{X}_{f(A)}$. Since f is fuzzy open, it follows that $y_r\in(\Im_2)_{f(A)}$. ii) Since P is σ-semi-discrete, $S(P)=\bigcup_{n=1}^{\infty}A_n$ with $(A_n,\Im_{A_n})$ is semi-discrete for all $n\in\mathbb{N}$. By Proposition 2.14 (i), $S(f(P))=f(S(P))=f(\bigcup_{n=1}^{\infty}A_n)=\bigcup_{n=1}^{\infty}f(A_n)$. Also, by (i) we have $(f(A_n),(\Im_2)_{f(A_n)})$ is semi-discrete for all $n\in\mathbb{N}$. It follows that $f(P)$ is σ-semi-discrete.

Theorem 3.10. In fuzzy topological spaces, "Being "DH" is a fuzzy topological property. Proof. Assume $(X,\Im_1)$ is a DH fuzzy topological space and let $f:(X,\Im_1)\to(Y,\Im_2)$ be a fuzzy homeomorphism where $(Y,\Im_2)$ is a fuzzy topological space. Choose a σ-semi-discrete dense(I) collection of fuzzy points P of $(X,\Im_1)$. According to Lemma 3.9 (ii) and Proposition 2.14 (ii), $f(P)$ will be σ-semi-discrete dense(I) in $(Y,\Im_2)$. Let P and W be any two σ-semi-discrete dense(I) collections of fuzzy points of $(Y,\Im_2)$. Then by Lemma 3.9 (ii) and Proposition 2.14 (ii), $f^{-1}(P)$ and $f^{-1}(W)$ are two σ-semi-discrete dense(I) collections of fuzzy points of $(X,\Im_1)$. Since $(X,\Im_1)$ is DH, there is a fuzzy homeomorphism $h:(X,\Im_1)\to(X,\Im_1)$ such that $h(S(f^{-1}(P)))=S(f^{-1}(W))$. Define $g:(Y,\Im_2)\to(Y,\Im_2)$ by $g=f\circ h\circ f^{-1}$. Then g is a fuzzy homeomorphism. Using Proposition 2.14 (i), we can see that $g(S(P))=S(W)$.

4. RELATIONSHIPS BETWEEN DH AND CDH FUZZY TOPOLOGICAL SPACES

In this section, we will give some relationships between DH and CDH fuzzy topological spaces. The following useful lemma follows easily: Lemma 4.1. Let $(X,\Im)$ be a fuzzy topological space and P be a collection of fuzzy points of X with S(P) is countable and non-empty. Then P is σ-semi-discrete. Theorem 4.2. Let $(X,\Im)$ be a fuzzy topological space for which X is countable. Then $(X,\Im)$ is DH iff $(X,\Im)$ is semi-discrete. Proof. Since the result is obvious when $|X|=1$, we will assume that $|X|>1$. Suppose that $(X,\Im)$ is DH and assume on the contrary that $(X,\Im)$ is not semi-discrete. Then there exists $y\in X$ such that $y_a\notin\Im$ for all $0<a\le 1$. Set $P=\mathbb{Q}(X)$ and $W=\mathbb{Q}(X\setminus\{y\})$. It is not difficult to see that P and W are dense (I). Also, by Lemma 4.1, P and W are σ-semi-discrete. So there is a fuzzy homeomorphism $h:(X,\Im)\to(X,\Im)$ such that $h(S(P))=S(W)$, therefore, $h(X)=X\setminus\{y\}$ which is a contradiction since h is an onto map.

Conversely, suppose that $(X,\Im)$ is semi-discrete. Then by Proposition 2.15 (ii), $(X,\Im)$ is separable. Choose a countable dense (I) collection of fuzzy points P. Then S(P) is countable and by Lemma 4.1, P is σ-semi-discrete. Let P and W be any two σ-semi-discrete dense(I) collections of fuzzy points. Then by Proposition 2.15 (i), $S(P)=S(W)=X$ and the identity fuzzy map completes the proof. Corollary 4.3. Let $(X,\Im)$ be a fuzzy topological space for which X is countable. Then $(X,\Im)$ is CDH iff $(X,\Im)$ is DH. Proof. Follows from Proposition 2.16 and Theorem 4.2. Theorem 4.4. If $(X,\Im)$ is separable and DH fuzzy topological space, then $(X,\Im)$ is CDH. Proof. Follows from the definitions and Lemma 4.1.

Recall that a fuzzy topological space $(X,\Im)$ is hereditarily separable if every subspace of $(X,\Im)$ is separable. Recall that a fuzzy topological space is second countable if it has a countable base. It is well known that second countable fuzzy topological spaces are hereditarily separable. Lemma 4.5. If $(X,\Im)$ is a hereditarily separable fuzzy topological space and P is a σ-semi-discrete collection of fuzzy points of $(X,\Im)$, then $S(P)$ is countable. Proof. Since P is σ-semi-discrete, then $S(P) = \bigcup_{n=1}^{\infty} A_n$ with $(A_n, \mathfrak{I}_{A_n})$ is semi-discrete for all $n \in \mathbb{N}$. Since $(X, \mathfrak{I})$ is hereditarily separable, then for each $n \in \mathbb{N}$, $(A_n, \mathfrak{I}_{A_n})$is separable and by Proposition 2.15 (ii) it follows that A_n is countable. Thus, $S(P)$ is countable.

Theorem 4.6. If $(X, \mathfrak{I})$ is hereditarily separable and CDH fuzzy topological space, then $(X, \mathfrak{I})$ is DH. Proof. Since $(X, \mathfrak{I})$ is hereditarily separable, then it is separable. So, there exists a countable dense (I) collection of fuzzy points P and by Lemma 4.1, P is σ-semi-discrete. Let P and W be two σ-semi-discrete dense(I) collections of fuzzy points. Then by Lemma 4.5, $S(P)$ and $S(W)$ are countable. By Proposition 2.11, $\mathbb{Q}(S(P))$ and $\mathbb{Q}(S(W))$ are countable dense(I). Since $(X, \mathfrak{I})$ is CDH, there is a fuzzy homeomorphism $h: (X, \mathfrak{I}) \rightarrow (X, \mathfrak{I})$ such that $h(S(P)) = h(S(\mathbb{Q}(S(P)))) = S(\mathbb{Q}(S(W))) = S(W)$. Corollary 4.7. Let $(X, \mathfrak{I})$ be a hereditarily separable fuzzy topological space. Then $(X, \mathfrak{I})$ is CDH iff $(X, \mathfrak{I})$ is DH. Proof. Follows from Theorems 4.4 and 4.6. Corollary 4.8. Let $(X, \mathfrak{I})$ be a second countable fuzzy topological space. Then $(X, \mathfrak{I})$ is CDH iff $(X, \mathfrak{I})$ is DH.

5. CUT TOPOLOGICAL SPACES

In this section we will mainly show that a-cut topological space $(X, \mathfrak{I}_a)$ of a fuzzy topological $(X, \mathfrak{I})$ is DH in general only if $a = 0$. Lemma 5.1. Let $(X, \mathfrak{I})$ be a fuzzy topological space. Let B be non-empty subset of X and let P be a collection of fuzzy points of X. Then

- $(B, \mathfrak{I}_B)$ is semi-discrete iff $(\mathfrak{I}_0)_B$ is the discrete topology on B.
- If B is σ-discrete in $(X, \mathfrak{I}_0)$, then $\mathbb{Q}(B)$ is σ-semi-discrete in $(X, \mathfrak{I})$.
- If P is σ-semi-discrete in $(X, \mathfrak{I})$, then $S(P)$ is σ-discrete in $(X, \mathfrak{I}_0)$.

Proof. (i) Suppose that $(B, \mathfrak{I}_B)$ is semi-discrete and let $x \in B$. Then there exists a fuzzy point or a fuzzy crisp point x_a for some a with $x_a \in \mathfrak{I}_B$. Choose $\lambda \in \mathfrak{I}$ such that$x_a = \lambda \cap \mathcal{X}_B$. Then $(\lambda \cap \mathcal{X}_B)(x) = \min\{\lambda(x), \mathcal{X}_B(x)\} > 0$ and so, $\{x\} = \lambda^{-1}(0,1] \cap B \in (\mathfrak{I}_0)_B$. Conversely, suppose that $(\mathfrak{I}_0)_B$ is the discrete topology on B and let $x \in B$. Then there exists $\lambda \in \mathfrak{I}$ such that $\{x\} = \lambda^{-1}(0,1] \cap B$. Now, $\lambda \cap \mathcal{X}_B$ is the fuzzy or crisp point $x_{\lambda(x)}$, on the other hand, $\lambda \cap \mathcal{X}_B \in \mathfrak{I}_{B.}$ ii) Since B is σ-discrete in $(X, \mathfrak{I}_0)$, then $\bigcup_{n=1}^{\infty} B_n$ with $(\mathfrak{I}_0)_{B_n}$ is the discrete topology for all $n \in \mathbb{N}$. By (i), $\mathfrak{I}_{B_n}$ is semi-discrete for all $n \in \mathbb{N}$. Since $S(\mathbb{Q}(B)) = B$, then $\mathbb{Q}(B)$ is σ-discrete in $(X, \mathfrak{I})$. iii) Since P is σ-discrete in $(X, \mathfrak{I})$, then $S(P) = \bigcup_{n=1}^{\infty} A_n$ with $\mathfrak{I}_{A_n}$ is semi-discrete for all $n \in \mathbb{N}$. By (i), $(\mathfrak{I}_0)_{A_n}$ is the discrete topology on A_n for all $n \in \mathbb{N}$. It follows that $S(P)$ is σ-discrete in $(X, \mathfrak{I}_0)$. Theorem 5.2. If $(X, \mathfrak{I})$ is a DH fuzzy topological space, then $(X, \mathfrak{I}_0)$ is DH.

Proof. Suppose that $(X, \mathfrak{I})$ is DH. Then $(X, \mathfrak{I})$ has a σ-semi-discrete dense(I) collection of fuzzy points P. By Lemma 5.1 (iii) and Proposition 2.18 (ii), $S(P)$ is σ-discrete dense in $(X, \mathfrak{I}_0)$. Let A and B be any two σ-discrete dense sets in $(X, \mathfrak{I}_0)$. Then by Lemma 5.1 (ii) and Proposition 2.18 (i), $\mathbb{Q}(A)$ and $\mathbb{Q}(B)$ are σ-semi-discrete dense (I) in $(X, \mathfrak{I})$. Since $(X, \mathfrak{I})$ is DH, there exists a fuzzy homeomorphism $h: (X, \mathfrak{I}) \rightarrow (X, \mathfrak{I})$ such that $h(S(\mathbb{Q}(A))) = S(\mathbb{Q}((B)))$. By Proposition 2.17, $h: (X, \mathfrak{I}_0) \rightarrow (X, \mathfrak{I}_0)$ is a homeomorphism. On the other hand, $S(\mathbb{Q}(A)) = A$ and $S(\mathbb{Q}(B)) = B$. Therefore, $(X, \mathfrak{I}_0)$ is DH. The following proposition is well known:

Proposition 5.3. Let (X, τ) be a topological space with X is countable. Then the following are equivalent:

- (X, τ) is CDH.
- τ is the discrete topology on X.
- (X, τ) is DH.

Theorem 5.4. Let X be a countable set and let $(X, \mathfrak{I})$ be a fuzzy topological space. Then the following are equivalent:

- $(X, \mathfrak{I})$ is DH.
- $(X, \mathfrak{I})$ is CDH.
- $(X, \mathfrak{I}_0)$ is DH.
- $(X, \mathfrak{I}_0)$ is CDH.

Proof. Follows from Theorem 4.2, Lemma 5.1 (i) and Proposition 5.3. In fact if $a > 0$, then $(X, \mathfrak{I})$ being DH does not imply, in general, that $(X, \mathfrak{I}_a)$ is DH. This will be explained in the following counterexample: Example 5.5. For fixed $0 < a < 1$, let $X = \{x, y\}$ and define $\mathfrak{I} = \{0, 1, x_{a/2}, y_{a/4}, x_{a/2} \cup y_{a/4}\}$. It is clear that $(X, \mathfrak{I})$ is semi-discrete and so by Theorem 4.2, it is DH. On the other hand, since $\mathfrak{I}_a = \{\emptyset, X\}$, then $(X, \mathfrak{I}_a)$ is not DH.

REFERENCES

[1] L. A. Zadeh, "Fuzzy Sets," *Inform and control*, vol. 8, pp. 338-353, 1965.
[2] C. L. Chang, "Fuzzy Topological Spaces," *Journal of Mathematical Analysis and Applications*, vol. 24, pp. 182-190, 1968.
[3] S. Al Ghour, "Homogeneity in fuzzy spaces and their induced spaces," *Questions and Answers in General Topology*, vol. 21, pp. 185-195, 2003.
[4] S. Al Ghour, "SLH fuzzy spaces," *African Diaspora Journal of Mathematics*, vol. 2, pp. 61-67, 2004.
[5] S. Al Ghour, A. Fora, "Minimality and Homogeneity in Fuzzy Spaces," *Journal of Fuzzy Mathematics*, vol. 12, pp. 725--737, 2004.
[6] S. Al Ghour, "Local homogeneity in fuzzy topological spaces," *International Journal of Mathematics and Mathematical Sciences*, Art. ID 81497, 14 pp, 2006.
[7] S. Al Ghour, "Some Generalizations of Minimal Fuzzy Open Sets," *Acta Mathematica Universitatis Comenianae*, vol. 75, pp. 107-117, 2006.
[8] S. Al Ghour, K. Al-Zoubi, "On some ordinary and fuzzy homogeneity types," *Acta Mathematica Universitatis Comenianae*, vol. 77, pp. 199-208, 2008.
[9] S. Al Ghour, A. Fora, "On CDH fuzzy spaces," *Journal of Intelligent & Fuzzy Systems*, vol. 30, pp. 935-941, 2016.
[10] A. Fora, S. Al Ghour, "Homogeneity in Fuzzy Spaces," *Questions and Answers in General Topology*, vol. 19, pp. 159-164, 2001.
[11] S. Al Ghour, A. Azaizeh, "Fuzzy Homogeneous Bitopological Spaces," *International Journal of Electrical and Computer Engineering*, vol. 8, pp. 2088-8708, 2018.
[12] R. Bennett, "Countable dense homogeneous spaces," Fundamenta Mathematicae, vol. 74, pp.189-194, 1972.
[13] A.V. Arhangel'skii, J. van Mill, "On the cardinality of countable dense homogeneous spaces," *Proceedings of the American Mathematical Society*, vol. 141, pp. 4031-4038, 2013.
[14] R. Hernandez-Gutiérrez, "Countable dense homogeneity and the double arrow space," *Topology and Its Applications*, vol. 160, pp. 1123-1128, 2013.
[15] R. Hernandez-Gutiérrez, M. Hrušak, "Non-meager P-filters are countable dense homogeneous," *Colloquium Mathematicum*, vol. 130, pp. 281-289, 2013.
[16] R. Hernandez-Gutiérrez, M. Hrušak, J. van Mill, "Countable dense homogeneity and λ-sets," *Fundamenta Mathematicae*, vol. 226, pp. 157-172, 2014.
[17] M. Hrusak, J. van Mill, "Nearly countable dense homogeneous spaces," *Canadian Journal of Mathematics*, vol. 66, pp. 743-758, 2014.
[18] J. van Mill, "On countable dense and n-homogeneity," *Canadian Mathematical Bulletin*, vol. 56, pp. 860-869, 2013.
[19] J. van Mill, "Countable dense homogeneous rimcompact spaces and local connectivity," *Filomat*, vol. 29, pp. 179-182, 2015.
[20] D. Repovš, L. Zdomskyy, S. Zhang, "Countable dense homogeneous filters and the Menger covering property," *Fundamenta Mathematicae*, vol. 224, pp. 233-240, 2014.
[21] B. Fitzpatrick, N. F. Lauer, "Densely homogeneous spaces (I)," *Houston Journal of Mathematics*, vol. 13, pp. 19-25, 1987.
[22] S. K. Cho, "Some results related to densely homogeneous spaces," *Communications of the Korean Mathematical Society*, vol. 11, pp. 1061-1066, 1996.
[23] D. L. Fearnley, "A Moore space with a σ-discrete π-base which cannot be densely embedded in any Moore space with the Baire property," *Proceedings of the American Mathematical Society*, vol. 127, pp. 3095-3100, 1999.
[24] B. Fitzpatrick, H. X. Zhou, "Densely homogeneous spaces. (II)," *Houston Journal of Mathematics*, vol. 14, pp. 57-68, 1988.
[25] B. Fitzpatrick, H. X. Zhou, "Some open problems in densely homogeneous spaces," *Open problems in topology, North-Holland, Amsterdam*, pp. 251-259, 1990.
[26] B. Fitzpatrick, J. White, H. X. Zhou, "Homogeneity and σ-discrete sets," *Topology and Its Applications*, vol. 44, pp. 143-147, 1992.
[27] S. V. Medvedev, "Metrizable DH-spaces of the first category," *Topology and Its Applications*, vol. 179, pp. 171-178, 2015.
[28] W. L. Saltsman, "Components of densely homogeneous spaces," *Houston Journal of Mathematics*, vol. 18, pp. 417-422, 1992.
[29] C. K. Wong, "Fuzzy points and local properties of fuzzy topology," *Journal of Mathematical Analysis and Applications*, vol. 46, pp. 316-328, 1974.
[30] R. Srivastava, S.N. Lal, A.K. Srivastava, "Fuzzy Hausdorff topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 81, pp. 497-506, 1981.
[31] M. H.Ghanim, E. E. Kerre, A. S. Mashhour, "Separation axioms, subspaces and sums in fuzzy topology," *Journal of Mathematical Analysis and Its Applications*, vol. 102, pp. 189-202, 1984.
[32] A. Fora, "Separation axioms for fuzzy spaces," *Fuzzy Sets and Systems*, vol. 33, pp. 59-75, 1989.
[33] R. Lowen, "A comparison of different compactness notions in fuzzy topological spaces," *Journal of Mathematical Analysis and Its Applications*, vol. 64, pp. 446-454, 1978.
[34] G. J. Wang, "Theory of L-fuzzy topological space," Shanxi Normal University Press, Xian, (in Chinese), 1988.
[35] A. Fora, "Separation axioms, subspaces and product spaces in fuzzy topology," *Arab Gulf Journal for Scientific Research*, vol. 8, pp. 1-16, 1990.