Efficient approximate analytical methods for nonlinear fuzzy boundary value problem

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ABSTRACT

This paper aims to solve the nonlinear two-point fuzzy boundary value problem (TPFBVP) using approximate analytical methods. Most fuzzy boundary value problems cannot be solved exactly or analytically. Even if the analytical solutions exist, they may be challenging to evaluate. Therefore, approximate analytical methods may be necessary to consider the solution. Hence, there is a need to formulate new, efficient, more accurate techniques. This is the focus of this study: two approximate analytical methods-homotopy perturbation method (HPM) and the variational iteration method (VIM) is proposed. Fuzzy set theory properties are presented to formulate these methods from crisp domain to fuzzy domain to find approximate solutions of nonlinear TPFBVP. The presented algorithms can express the solution as a convergent series form. A numerical comparison of the mean errors is made between the HPM and VIM. The results show that these methods are reliable and robust. However, the comparison reveals that VIM convergence is quicker and offers a swifter approach over HPM. Hence, VIM is considered a more efficient approach for nonlinear TPFBVPs.

Keywords:
Approximate analytical methods
Fuzzy differential equations
Homotopy perturbation method
Two-point BVP
Variational iteration method

1. INTRODUCTION

Real-life applications investigate the meaning of fuzzy as a generalization of crisp common sense because it is a solid instrument for modeling the vagueness, and in specific, to treat uncertainty with a mathematical model [1]. In many real well-determined dynamic issues, a system of ordinary or partial differential equations may represent the mathematical model. On the contrary, fuzzy differential equations (FDEs) are a valuable tool to model a dynamic system that is ambiguous in its existence and comportments. Since the FDEs have been used widely to model scientific and engineering problems, they have become a popular topic among researchers [2]. There are many practical problems with the solution of FDEs that satisfy initial [3] or boundary [4] values conditions. The main reason why finding the approximate solutions to the fuzzy problems becomes necessary is that most of the problems are too complicated to be solved exactly, or there are no analytical solutions at all [4]. Hence, FDEs will be suitable mathematical models for dynamic systems where complexity and ambiguity occur. For this reason, we may find FDEs exist in several fields of mathematics and science, including population models [5–7] and mathematical biology and
physics [8]–[10]. As an alternative to the analytical solutions to such problems, approximate solutions such as homotopy perturbation method (HPM) and variational iteration method (VIM), and Adomian decomposition method (ADM) are listed as some of the approximate analytical methods [11].

In the last decade, some researchers have started to explore the numerical solutions for two-point fuzzy boundary value problem (TPFBVP) [12], [13]. Semi-analytical approaches have been used over recent years to overcome linear TPFBVP by various methods [14]–[16]. He implemented the HPM in 1999, and the method was applied to a wide range of mathematical and physical problems [17], [18]. This method provides the solution into components of a short convergence series, which are elegantly determined. HPM is now known as a standard tool for overcoming all kinds of linear and nonlinear equations, such as differential or integral equations. Another significant advantage is that the measurement size can be reduced while increasing the exactness of approximate solutions, so it is regarded as a robust process. Along with the HPM, VIM was introduced and also proposed by He [19]. This approach differs from specific classical techniques utilizing which nonlinear equations are quickly and accurately resolved. VIM has been used in many physics and engineering sectors recently [20], [21]. This approach is helpful for directly solving linear and nonlinear problems with $n^{th}$ order boundary value problems (BVPs) without reducing them to a BVP system. It has been reported by many authors, such as [22], that VIM is more robust than other analytical approaches, like ADM and HPM. Compared to HPM and ADM, where computer algorithms are commonly used for nonlinear terms, VIM is used explicitly without any nonlinear terms requirement or restrictive assumptions [23]. Without restrictive assumptions, the VIM solves differential equations that can change the structure of solutions. In VIM, the calculation is simple and straightforward [24]. The VIM overcomes the difficulty of measuring Adomian polynomials [23], a significant advantage over ADM.

According to [24], one significant disadvantage of VIM is that the terms obtained are longer than those obtained by decomposition and perturbation methods. For this reason, we are seeking and investigating the proposed approximate analytical solutions for nonlinear FDEs by HPM and VIM for comparison purposes. This study will develop an innovative approach to modifying the nonlinear TPFBVP based on the framework of fuzzy problems. This modification is tested on two existing FDEs and compared with the exact solution and the numerical solution. A comparative study will be given to show the capabilities of the proposed methods. According to the results, the modified schemes were found to be feasible and more accurate.

2. PRELIMINARIES

In this section, we provide some fundamental concepts and definitions that are necessary for this work. This includes some propositions, properties, and explanations of fuzzy sets and numbers and FDEs that will be used later in this work.

Definition 2.1 [25]: the relation:

$$
\mu(x;\alpha,\beta,\gamma) = \begin{cases} 
0, & \text{if } x < \alpha \\
\frac{x - \alpha}{\beta - \alpha}, & \text{if } \alpha \leq x \leq \beta \\
\frac{\gamma - x}{\gamma - \beta}, & \text{if } \beta \leq x \leq \gamma \\
0, & \text{if } x > \gamma
\end{cases}
$$

is a form of the membership function for a trapezoidal fuzzy number $\mu = (x;\alpha,\beta,\gamma)$, which is presented as shown in Figure 1.

![Figure 1. Triangular fuzzy number](image-url)
And its r-level is: \( [\mu]_r = [\alpha + r (\beta - \alpha), \gamma - r (\gamma - \beta)] \), \( r \in [0, 1] \).

Definition 2.2 [25]: The relation

\[
\mu(x; \alpha, \beta, \gamma, \delta) = \begin{cases} 
  0, & \text{if } x < \alpha \\
  \frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \leq x \leq \beta \\
  1, & \text{if } \beta \leq x \leq \gamma \\
  \frac{\delta-x}{\delta-\gamma}, & \text{if } \gamma \leq x \leq \delta \\
  0, & \text{if } x > \delta 
\end{cases}
\]

Is a form of the membership function for a trapezoidal fuzzy number \( \mu = (x; \alpha, \beta, \gamma, \delta) \), which is presented as in Figure 2.

![Figure 2. Trapezoidal fuzzy number](image)

This can be used to describe an r-level set of the trapezoidal fugitive number as:

\[
[\tilde{\mu}]_r = ([\beta - \alpha]r + \alpha, [\delta - \gamma]r + \gamma]
\]

This paper describes the class of all fuzzy subsets of \( \mathbb{R} \) is being marked by \( \tilde{E} \) that satisfies the characteristics fuzzy number properties [26].

Definition 2.3 [27]: Let \( \tilde{f}: \mathbb{R} \to \tilde{E}, \tilde{f}(x) \) is called fuzzy function if \( \tilde{E} \) is a set of fuzzy numbers.

Definition 2.4 [28]: The r-level set defined as \( [\tilde{f}(x)]_r = [\tilde{f}(x); \tilde{f}(x)], x \in K, r \in [0,1] \) for a fuzzy function \( \tilde{f}: T \to \tilde{E} \) where \( T \subseteq \tilde{E} \). A fuzzy number is more effective than the r-level sets as representational types of fuzzy sets. Fuzzy sets can be described based on the resolution identity theorem by the families in their r-level sets.

Definition 2.5 [29]: If \( f: X \to Y \) is function induces another function \( \tilde{f}: F(X) \to F(Y) \) For each interval, \( U \) in \( X \) is defined by:

\[
\tilde{f}(U)(y) = \begin{cases} 
  \text{Sup}_{x \in f^{-1}(y)} U(x), & \text{if } y \in \text{range} (f) \\
  0, & \text{if } y \notin \text{range} (f)
\end{cases}
\]

This is recognized as the theory of Zadeh extension principle.

Definition 2.6 [30]: Let \( \tilde{z} = \tilde{x} \ominus \tilde{y} \) be the H-difference of the fuzzy numbers \( \tilde{x} \) and \( \tilde{y} \) if the fuzzy number \( \tilde{z} \) exist with the property \( \tilde{z} = \tilde{y} + \tilde{z} \).

Definition 2.7 [13]: If \( \tilde{f}: I \to \tilde{E} \) and \( t_0 \in I \), where \( I \subseteq [a, b] \), \( \tilde{f}' \) is said to be Hukuhara differentiable at \( t_0 \), if there exists an element \( [\tilde{f}]_r \in \tilde{E} \) to be small enough for all \( h > 0 \) (near to 0), exists \( \tilde{f}(t_0 + h; r) \ominus \tilde{f}(t_0; r) \ominus \tilde{f}(t_0 - h; r) \) and limits in metric(\( \tilde{E} \), \( D \)) are taken and exist in such a way that

\[
\tilde{f}'(t_0) = \lim_{h \to 0^+} \frac{[\tilde{f}(t_0 + h; r) \ominus \tilde{f}(t_0; r)]}{h}
\]

For more details, see [30].
Definition 2.8 [31]: Settle $\tilde{f}: I \rightarrow \tilde{E}$ and $t_0 \in I$, for $\in [a, b]$. $\tilde{f}^{(n)}$ is said to be Hukuhara differentiable $x \in \tilde{E}$, if there exists an element $[\tilde{f}^{(n)}]_r \in \tilde{E}$ to be small enough for all $h > 0$ (near to 0), exists $\tilde{f}^{(n-1)}(t_0 + h; r) \odot \tilde{f}^{(n)}(t_0; r), \tilde{f}^{(n-1)}(t_0; r) \odot \tilde{f}^{(n-1)}(t_0 - h; r)$ and limits in metric $(\tilde{E}, D)$ are taken and exist in such a way that

$$\tilde{f}^{(n)}(t_0) = \lim_{h \to 0^{+}} \frac{\tilde{f}^{(n-1)}(t_0 + h; r) \odot \tilde{f}^{(n)}(t_0; r) - \tilde{f}^{(n-1)}(t_0; r) \odot \tilde{f}^{(n-1)}(t_0 - h; r)}{h}$$

There is a second order of the derivatives of Hukuhara for $n = 2$ and equivalent to $\tilde{f}^{(n)}$.

Theorem 2.1 [31]: Let $\tilde{f}: [t + a, b] \rightarrow \tilde{E}$ be Hukuhara differentiable and denote

$$[\tilde{F}^{(r)}(t)]_r = \begin{bmatrix} \tilde{f}'(t), \tilde{f}''(t) \end{bmatrix}_r = \begin{bmatrix} \tilde{f}'(t; r), \tilde{f}''(t; r) \end{bmatrix}_r$$

Then we can define the differentiable boundary functions $\tilde{f}^{(r; t); n}$ and $\tilde{f}^{(t; r); n}$ can be written in the $n^{th}$ order of $n^{th}$ fuzzy derivatives.

$$[\tilde{F}^{(r; t); n}(t; r)]_r = \begin{bmatrix} \tilde{f}^{(r; t); n}(t; r) \end{bmatrix}_r$$

∀ $r \in [0, 1]$  

3. DESCRIPTION OF THE FUZZY HPM

The overall HPM structure for solving crisp nonlinear TFBVP is mentioned in [17], [18]. Consider the defuzzification of the following general $n^{th}$ order TFBVP [15].

$$\tilde{y}^{(n)}(t) = f(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \ldots , \tilde{y}^{(n-1)}(t) + \tilde{g}(t), \ t \in [t_0, T]$$

$$\tilde{y}(t_0) = \tilde{a}^{(0)}, \tilde{y}'(t_0) = \tilde{a}^{(1)}, \ldots , \tilde{y}^{(k)}(t_0) = \tilde{a}^{(k)}, \tilde{y}^{(n)}(T) = \tilde{b}^{(n-k-2)}(T) = \tilde{b}^{(n-k-2)}$$

To solve (1) by using HPM, we need to fuzzify HPM and then defuzzify it back to (1) as in [15]. According to [32], the HPM and for all $r \in [0, 1]$. HPM must ensure the convergence of the HPM solution series function through the correct choice of initial guess and the auxiliary linear operator [33]. From [15], HPM form for solving (1) is given by solving the lower bound as follows:

$$\begin{align*}
\text{p}^0: & \quad \mathcal{L}_n \left[ y_0(t; r) - y_0(t; r; \sum_{i=1}^{n} \mathcal{L}_r \mathcal{C}_r (r)) \right] = 0, \\
& \quad y(t_0; r) = \left[ \begin{array}{c} \mathcal{L}_r \mathcal{C}_r (r) \end{array} \right], y'(t_0; r) = \left[ \begin{array}{c} \mathcal{L}_r \mathcal{C}_r (r) \end{array} \right], \ldots , y^{(n-1)}(t_0; r) = \left[ \begin{array}{c} \mathcal{L}_r \mathcal{C}_r (r) \end{array} \right], \\
& \quad y(T; r) = \left[ \begin{array}{c} \mathcal{L}_r \mathcal{C}_r (r) \end{array} \right], y'(T; r) = \left[ \begin{array}{c} \mathcal{L}_r \mathcal{C}_r (r) \end{array} \right], \ldots , y^{(n-k-2)}(T; r) = \left[ \begin{array}{c} \mathcal{L}_r \mathcal{C}_r (r) \end{array} \right].
\end{align*}$$

$$\begin{align*}
\text{p}^1: & \quad \mathcal{L}_n \left[ y_1(t; r) + y_0(t; r; \sum_{i=1}^{n} \mathcal{L}_r \mathcal{C}_r (r)) \right] - g(t, \tilde{y}(t; r; \sum_{i=1}^{n} \mathcal{L}_r \mathcal{C}_r (r))) \\
& \quad y_1(t_0; r) = 0, y'_1(t_0; r) = 0, \ldots , y^{(n-1)}_1(t_0; r) = 0, \\
& \quad y_1(T; r) = 0, y'_1(T; r) = 0, \ldots , y^{(n-k-2)}_1(T; r) = 0.
\end{align*}$$

$$\begin{align*}
\text{p}^2: & \quad \mathcal{L}_n y_2(t; r) - F(t, \tilde{y}_1(t; r; \sum_{i=1}^{n} \mathcal{L}_r \mathcal{C}_r (r))) = 0, \\
& \quad y_2(t_0; r) = y_0(t_0; r) = 0, \ldots , y^{(n-1)}_2(t_0; r) = 0, \\
& \quad y_2(T; r) = y_2(T; r) = 0, \ldots , y^{(n-k-2)}_2(T; r) = 0.
\end{align*}$$

$$\vdots$$

$$\begin{align*}
\text{p}^{n+1}: & \quad \mathcal{L}_n y_{n+1}(t; r) - F(t, \tilde{y}_n(t; r; \sum_{i=1}^{n} \mathcal{L}_r \mathcal{C}_r (r))) = 0, \\
& \quad y_{n+1}(t_0; r) = y_0(t_0; r) = 0, \ldots , y^{(n-1)}_{n+1}(t_0; r) = 0, \\
& \quad y_{n+1}(T; r) = y_{n+1}(T; r) = 0, \ldots , y^{(n-k-2)}_{n+1}(T; r) = 0.
\end{align*}$$

Efficient approximate analytical methods for nonlinear fuzzy boundary value problem (Ali Fareed Jameel)
Similarly for the upper bound

\[
\begin{align*}
p^0: & \quad \bar{L}_n[\bar{y}(t; r) - \bar{y}_0(t; r; \sum_{i=1}^{n} \bar{e}_s(r))] = 0, \\
p^1: & \quad \bar{y}_1(t_0; r) = 0, \bar{y}_1' (t_0; r) = 0, ..., \bar{y}_1^{(n-1)} (t_0; r) = 0, \\
p^2: & \quad \bar{y}_2(t_0; r) = 0, \bar{y}_2' (t_0; r) = 0, ..., \bar{y}_2^{(n-1)} (t_0; r) = 0, \\
\vdots
\end{align*}
\]

where \( \bar{y}(t) = \bar{y}(t), \bar{y}'(t), \bar{y}''(t), ..., \bar{y}^{(n-1)}(t) \) and the initials guessing for the approximate solution is given in [16] for all \( r \in [0,1] \) and then the exact solution of (1) can now be obtained by setting \( p = 1 \) as in (3):

\[
\bar{y}(t; r; \sum_{i=1}^{n} \bar{e}_s(r)) = S_m(t; r; \sum_{i=1}^{n} \bar{e}_s(r)) = \sum_{i=0}^{m-1} \bar{y}_i(t; r; \sum_{i=1}^{n} \bar{e}_s(r))
\]

Therefore, the exact solution of (1) can now be obtained by setting \( p = 1 \):

\[
\bar{y}(t; r; \sum_{i=1}^{n} \bar{e}_s(r)) = \lim_{p \to 1} \bar{y}(t; r; \sum_{i=1}^{n} \bar{e}_s(r)) = \lim_{p \to 1} \sum_{i=0}^{m-1} \bar{y}_i(t; r; \sum_{i=1}^{n} \bar{e}_s(r))
\]

\[\text{(4)}\]

4. DESCRIPTION OF THE FUZZY VIM

VIM general structure for solving problems TFPBVP is stated in [19]. In order for the (1) to be solved using VIM, we must fuzzify and defuzzify the VIM as defined in (2) [34]. According to VIM in [22] and for all \( r \in [0,1] \) we rewrite (1) in the following correction functional forms:

\[
\begin{align*}
\frac{y_i}{t+1}(t; r; \sum_{i=1}^{n} e_s(r)) &= \frac{y_i}{t+1}(t; r; \sum_{i=1}^{n} e_s(r)) + \\
\int_{\eta}^{t} \lambda (t; \eta) \left\{ \frac{y_i}{t+1}(\eta; r) + F \left( \eta, \bar{y}(\eta; r) \right) + g(\eta; r) \right\} d\eta,
\end{align*}
\]

\[\text{(5)}\]

\[
\begin{align*}
\bar{y}_i(t; r; \sum_{i=1}^{n} e_s(r)) &= \bar{y}(t; r; \sum_{i=1}^{n} e_s(r)) + \int_{\eta}^{t} \lambda (t; \eta) \left\{ \bar{y}_i^{(n)}(\eta; r) + G \left( \eta, \bar{y}(\eta; r) \right) + g(\eta; r) \right\} d\eta,
\end{align*}
\]

\[\text{(6)}\]

where \( i = 1, 2, ..., r \in [0,1] \). The Lagrange multiplier is \( \lambda(t; \eta) \) which can be optimally defined via variational theory [33]. Now we let

\[
\begin{align*}
F, \bar{y}_i &= F \left( \eta, \bar{y}(\eta; r) \right) + g(\eta; r), \\
G, \bar{y}_i &= G \left( \eta, \bar{y}(\eta; r) \right) + \bar{g}(\eta; r).
\end{align*}
\]
where \( F \) and \( G \) are nonlinear operators including the nonlinear terms \( F \) and \( G \) and the inhomogeneous term \( \tilde{g}(\eta; r) \). In the following forms we will rewrite (5) and (6) as:

\[
\begin{align*}
\bar{y}_{i+1}(t; r; \sum_{s=1}^{n-1} \xi_s(r)) &= \bar{y}_i(t; r; \sum_{s=1}^{n-1} \xi_s(r)) + \int_0^t \lambda(t; \eta) \left\{ \bar{y}_i^{(n)}(\eta; r) + F(\eta; r) \right\} d\eta, \\
\tilde{y}_{i+1}(t; r; \sum_{s=1}^{n-1} \xi_s(r)) &= \tilde{y}_i(t; r; \sum_{s=1}^{n-1} \xi_s(r)) + \int_0^t \lambda(t; \eta) \left\{ \tilde{y}_i^{(n)}(\eta; r) + G(\eta; r) \right\} d\eta,
\end{align*}
\]

where restricted variation is \( \tilde{\gamma} \), i.e. \( \delta \tilde{\gamma} = 0 \) [21]. The general multiplier \( \lambda(t; \eta) \) applied to (1) according to [23] can be described in the following:

\[
\begin{align*}
\delta \bar{y}_{i+1}(t; r; \sum_{s=1}^{n-1} \xi_s(r)) &= \delta \bar{y}_i(t; r; \sum_{s=1}^{n-1} \xi_s(r)) + \delta \int_0^t \lambda(t; \eta) \left\{ \bar{y}_i^{(n)}(\eta; r) + F(\eta; r) \right\} d\eta \\
\delta \tilde{y}_{i+1}(t; r; \sum_{s=1}^{n-1} \xi_s(r)) &= \delta \tilde{y}_i(t; r; \sum_{s=1}^{n-1} \xi_s(r)) + \delta \int_0^t \lambda(t; \eta) \left\{ \tilde{y}_i^{(n)}(\eta; r) \right\} d\eta.
\end{align*}
\]

According to [23], we obtain the followings by integrating by part:

\[
\begin{align*}
\bar{y}_{i+1}(t; r; \sum_{s=1}^{n-1} \xi_s(r)) &= \left[ 1 - \lambda(t)^{(n-1)} \right] \bar{y}_i(t; r; \sum_{s=1}^{n-1} \xi_s(r)) + \sum_{k=2}^{n-2} \delta \bar{y}_i^{(n-k)}(t; r) + \\
&+ \int_0^t \lambda(t; \eta)^{(n)} \delta \bar{y}_i(t; r; \sum_{s=1}^{n-1} \xi_s(r)) d\eta.
\end{align*}
\]

The following stationary conditions can therefore be reached [35].

\[
\begin{align*}
\lambda(\eta)^{(n)} &= 0, \\
1 - \lambda(t)^{(n-1)} &= 0, \\
\lambda(t)^{(k)} &= 0, \ for \ k = 0, 1, 2, ..., n - 2.
\end{align*}
\]

Under these terms and the order of the (1) the general Lagrangian multiplier can be calculated as [35]:

\[
\lambda(t; \eta) = \frac{(-1)^{n}(\eta-\tau)^{n-1}}{(n-1)!}
\]

Therefore, all the above parameters \( \lambda(t; \eta) \) and \( \bar{y}_0(t; r; \sum_{s=1}^{n} \xi_s(r)) \) will be easily obtained in the series of approximations of VIM. The exact solution can therefore be obtained:

\[
\bar{y}(t; r) = \lim_{i \to \infty} \bar{y}_i(t; r; \sum_{s=1}^{n} \xi_s(r)).
\]

To determine the \( \xi_s(r) \), we use the same HPM technique as in [15] by substituting these constants with the initial estimates in series solution function and then using the boundary conditions of (1) to determine the values of these constants for each fuzzy \( r \)-level set.

5. NUMERICAL EXAMPLES

The approximate solution by HPM and VIM in the next examples are obtained by formulating the given equations as presented in sections 3 and 4. The formula is then solved and analyzed by using Mathematica 11:

**Example 5.1:** Let us consider the following nonlinear second order TPFBVP:

\[
y''(t) + \bar{y}^2(t) = t^4 + 2, \ \bar{y}(0) = \bar{a}, \bar{y}(1) = \bar{b}
\]

where \( \bar{a} \) and \( \bar{b} \) are triangular fuzzy numbers having \( r \)-level sets \([0.1r - 0.1, 0.1 - 0.1r] \) and \([0.9 + 0.1r, 1.1 - 0.1r] \) for all \( r \in [0, 1] \). The linear operator in compliance with section 2 is \( \bar{L}_2 = \frac{d^2}{d\xi^2} \) with the inverse operator \( \bar{L}_2^{-1} \), and the initial guesses for all \( r \in [0, 1] \) are given by:

\[
\begin{align*}
\bar{y}_0(t; r) &= \xi_1(r) + \xi_2(r)t, \\
\bar{y}_0(t; r) &= \tilde{\xi}_1(r) + \tilde{\xi}_2(r)t.
\end{align*}
\]
5.1. HPM formulation

According to section 3, the values of \( \hat{c}_1(\tau) = [0.1r - 0.1, 0.1 - 0.1r] \) and the homotopy functions of (7) are

\[
H(t, r; \tau) = (1 - p)L_2 \left[ y(t; r) - y_0(t; r; \tilde{c}_2(\tau)) \right] + p \left[ L_2 y(t; r) + \left( y(t; r) \right)^2 - (t^4 + 2) \right] = 0,
\]

\[
H(t, r; \tau) = (1 - p)L_2 \left[ y(t; r) - y_0(t; r; \tilde{c}_2(\tau)) \right] + p \left[ L_2 y(t; r) + \left( y(t; r) \right)^2 - (t^4 + 2) \right] = 0,
\]

\[
H(t, r; \tau) = (1 - p)L_2 \left[ y(t; r) - y_0(t; r; \tilde{c}_2(\tau)) \right] + p \left[ L_2 y(t; r) + \left( y(t; r) \right)^2 - (t^4 + 2) \right] = 0.
\]

The \( n \) components are specified as the HPM in section 3 of \( \hat{y}_k(t; r) \) for \( k = 1, 2, \ldots, n \) and \( r \in [0, 1] \) calculated by evaluating the lower limit as follows:

\[
p^0: \left\{ \begin{array}{l}
y_0(t; r) = 0.1r - 0.1 + \tilde{c}_2(r)t, \\
y_1(t; r) = L_2^{-1} \left[ -y_0(t; r; \tilde{c}_2(r)) \right]^2 + t^4 + 2, \\
y_2(0; r) = 0, y_2(1; r) = 0.
\end{array} \right.
\]

\[
p^1: \left\{ \begin{array}{l}
y_0(t; r) = 0.1r - 0.1 + \tilde{c}_2(r)t, \\
y_1(t; r) = L_2^{-1} \left[ -y_0(t; r; \tilde{c}_2(r)) \right]^2 + t^4 + 2, \\
y_2(0; r) = 0, y_2(1; r) = 0.
\end{array} \right.
\]

For the upper limit, the values are obtained in the same way as follows:

\[
p^0: \left\{ \begin{array}{l}
\bar{y}_0(t; r) = 0.1r - 0.1 + \tilde{c}_2(r)t, \\
\bar{y}_1(t; r) = L_2^{-1} \left[ -y_0(t; r; \tilde{c}_2(r)) \right]^2 + t^4 + 2, \\
\bar{y}_2(0; r) = 0, \bar{y}_2(1; r) = 0.
\end{array} \right.
\]

\[
p^1: \left\{ \begin{array}{l}
\bar{y}_0(t; r) = 0.1r - 0.1 + \tilde{c}_2(r)t, \\
\bar{y}_1(t; r) = L_2^{-1} \left[ -y_0(t; r; \tilde{c}_2(r)) \right]^2 + t^4 + 2, \\
\bar{y}_2(0; r) = 0, \bar{y}_2(1; r) = 0.
\end{array} \right.
\]

Evaluating (8) and (9) to obtain fifth order HPM series solution in the following form such that

\[
\tilde{S}_2(t; r; \tilde{c}_2(r)) = \hat{y}_0(t; r; \tilde{c}_2(r)) + \sum_{k=1}^{5} \hat{y}_k(t; r; \tilde{c}_2(r)) = \tilde{y}(t; r; \tilde{c}_2(r))
\]

Now to obtain the values of \( \tilde{c}_2(r) \) for all \( r \in [0, 1] \), we solve the nonlinear series solution of (7) from the boundary condition \([0.9 + 0.1r, 1.1 - 0.1r]\) then we substitute the values of \( \tilde{c}_2(r) \) again in (10) to obtain fifth order HPM series solution. Since (7) is considered without exact analytical solution, to show the accuracy of fifth order HPM approximate series solution \( \tilde{S}_2(t; r) \) for all \( r \in [0, 1] \), the residual error must be specified:

\[
[\tilde{E}(t)]_r = [\tilde{S}_2(t; r)]''(t) + [\tilde{S}_2(t; r)]^2(t) - t^4 + 2.
\]

Then the fifth order HPM series solution is presented in the Tables 1-2 and Figure 3. According to Tables 1 and 2 and Figure 3, we concluded that the fifth order HPM approximate solutions of (7) for all \( t \in [0, 1] \) and \( r \in [0, 1] \) fulfill the patterns of fuzzy numbers in the form of a triangular fuzzy number.
Table 1. Approximate solution $y(t; r)$ of fifth order HPM at $t = 0.5$ and $r \in [0,1]$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$c_{5}(r)$</th>
<th>$y(0.5; r)$</th>
<th>$[E(0.5)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-0.012233517979781578$</td>
<td>$0.1436606833554236$</td>
<td>$4.183283991154862 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.25</td>
<td>$-0.010209834659421881$</td>
<td>$0.16997259607001605$</td>
<td>$2.94636275508149 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$-0.00750597713710612$</td>
<td>$0.19646322624295728$</td>
<td>$2.485028610110795 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.75</td>
<td>$-0.0041102080869923338$</td>
<td>$0.22313542853095245$</td>
<td>$2.726826934945636 \times 10^{-7}$</td>
</tr>
<tr>
<td>1</td>
<td>$-0.0001601865820840$</td>
<td>$0.2499204985486695$</td>
<td>$6.34210856364461 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Table 2. Approximate solution $\bar{y}(t; r)$ of fifth order HPM at $t = 0.5$ and $r \in [0,1]$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$c_{5}(r)$</th>
<th>$y(0.5; r)$</th>
<th>$[E(0.5)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$0.02353896960363461$</td>
<td>$0.35931908435924000$</td>
<td>$0.0000021162047742997$</td>
</tr>
<tr>
<td>0.25</td>
<td>$0.016554919244886100$</td>
<td>$0.3316966672670905$</td>
<td>$0.00000108129304682291$</td>
</tr>
<tr>
<td>0.5</td>
<td>$0.010307281584441760$</td>
<td>$0.30426986657956345$</td>
<td>$0.00000048897414700183$</td>
</tr>
<tr>
<td>0.75</td>
<td>$0.00478674073139446$</td>
<td>$0.2770359244404127$</td>
<td>$0.00000018877716657511$</td>
</tr>
<tr>
<td>1</td>
<td>$-0.00001601865820840$</td>
<td>$0.2499204985486695$</td>
<td>$6.34210856364461 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Figure 3. HPM approximate solution of (12) at $t = 0.5$ and $r \in [0,1]$

5.2. VIM formulation

The variational formula of this problem is given in accordance with section 4 as (11):

$$\begin{align*}
\bar{y}_{i+1} & = \bar{y}_{i} + \int_{0}^{\eta} \lambda(t; \eta) \left( \bar{y}_{i}''(\eta; r; \bar{c}_{2}(r)) + \bar{y}_{i}''(\eta; r; c_{2}(r)) \right) d\eta, \\
\bar{y}_{i+1} & = \bar{y}_{i} + \int_{0}^{\eta} \lambda(t; \eta) \left( \bar{y}_{i}''(\eta; r; \bar{c}_{2}(r)) + \bar{y}_{i}''(\eta; r; c_{2}(r)) \right) d\eta.
\end{align*}$$

(11)

The fourth-order VIM series solution is obtained in the form $12$:

$$\bar{s}_{4}(t; r; \bar{c}_{2}(r)) = \sum_{i=0}^{4} \bar{y}_{i}(t; r; \bar{c}_{2}(r)) = \bar{y}(t; r)$$

(12)

The Lagrangian multiplier of (11) is described in section 4 such that $\lambda(t; \eta) = \eta - t$. Now in order to obtain the values of $\bar{c}_{2}(r)$ for all $r \in [0,1]$ , we solve the nonlinear series solution of (7) from the boundary condition $[0.9 + 0.1r, 1.1 - 0.1r]$ then we substitute the values of $\bar{c}_{2}(r)$ again in (12) to obtain fourth-order VIM series solution. For (1), the following residual error is described in order to demonstrate VIM accuracy in approximate fourth-order solution without an exact analytical solution:

$$[\bar{E}(t)]_{r} = |\bar{s}_{4}''(t; r) + [\bar{s}_{4}(t; r)]^{2} - t^{4} - 2|$$

In Table 3 and 4 and Figure 4, the fourth order VIM series solution is presented:
Example 5.2 [35]: Consider this non-homogenous second-order non-linear TPFBVP:
\[
\begin{align*}
\dddot{y}(t) + \ddot{y}(t) &= \dddot{\tilde{y}}(t) + \ddot{\tilde{y}}(t) + \tilde{f}(t), t \in [0, 1] \\
\tilde{y}(0) &= \tilde{a}, \tilde{y}(1) &= \tilde{b}
\end{align*}
\]  
(13)

where \( \tilde{a} = \left[ \frac{1}{4} \left( \frac{r-2}{10} \right) + \frac{1}{4} \left( \frac{2-r}{10} \right) + \frac{1}{2} \right] \) and \( \tilde{b} = \left[ \frac{5}{4} \left( \frac{r-2}{10} \right) + \frac{1}{4} \left( \frac{r}{10} \right) + \frac{5}{4} \left( \frac{2-r}{10} \right) + \frac{e^{-1}}{4} \right] \) for all \( r \in [0, 1] \). According to [30] the fuzzy function \( \tilde{f}(t) \) have the following defuzzification:
\[
\tilde{f}(t) = \left[ \left( t + \frac{1}{4} \right) \left( \frac{r-2}{10} \right) - \frac{e^{-1}}{4} \right]^3 + \left( t + \frac{1}{4} \right) \left( \frac{2-r}{10} \right) - \frac{e^{-1}}{4} + \left( t + \frac{1}{4} \right) \left( \frac{r-2}{10} \right).
\]

Then the corresponding analytic solution of (13) is given by
\[
\tilde{y}(t; r) = \frac{e^{-t}}{4} + \left( t + \frac{1}{4} \right) \left( \frac{r-2}{10} \right) \left( \frac{2-r}{10} \right).
\]  
(14)

by following the initial guesses from Example 5.1, we have:
\[
\begin{align*}
\tilde{y}_0(t; r) &= \frac{1}{4} \left( \frac{r-2}{10} \right) + \frac{1}{4} + \tilde{y}_2(r)t, \\
\tilde{y}_0(t; r) &= \frac{1}{4} \left( \frac{r-2}{10} \right) + \frac{1}{4} + \tilde{y}_2(r)t.
\end{align*}
\]  
(15)
5.3. HPM formulation

From Section 4 the approximate solution of (13) can be determine by HPM from the followings:

\[
p^0: y_0(t; r) = \frac{1}{4} t^2 + \frac{1}{4} + c_2(r) t,
\]

\[
p^1: \begin{cases} 
  y_1(t; r) = L_2^{-1} \left[ y_0(t; r) + \left( y_0(t; r; c_2(r)) \right)^3 + f(t; r) \right], \\
  y_1(0; r) = 0, y_1(1; r) = 0.
\end{cases}
\]

\[
p^{k+1}: \begin{cases} 
  y_{k+1}(t; r) = L_2^{-1} y_k(t; r) + \sum_{i=0}^{k-1} \sum_{j=0}^{i} y_{k-1-i} \left( t; r; c_2(r) \right) y_j \left( t; r; c_2(r) \right) y_{i-j} \left( t; r; c_2(r) \right), \\
  y_k(0; r) = 0, y_k(1; r) = 0.
\end{cases}
\]

Similarly for the upper bound

\[
p^0: \bar{y}_0(t; r) = \frac{1}{4} (2t - r) + \frac{1}{4} + \bar{c}_2(r) t,
\]

\[
p^1: \begin{cases} 
  \bar{y}_1(t; r) = L_2^{-1} \left[ \bar{y}_0(t; r) + \left( \bar{y}_0(t; r; \bar{c}_2(r)) \right)^3 + \bar{f}(t; r) \right], \\
  \bar{y}_1(0; r) = 0, \bar{y}_1(1; r) = 0.
\end{cases}
\]

\[
p^{k+1}: \begin{cases} 
  \bar{y}_{k+1}(t; r) = L_2^{-1} \bar{y}_k(t; r) + \sum_{i=0}^{k-1} \sum_{j=0}^{i} \bar{y}_{k-1-i} \left( t; r; \bar{c}_2(r) \right) \bar{y}_j \left( t; r; \bar{c}_2(r) \right) \bar{y}_{i-j} \left( t; r; \bar{c}_2(r) \right), \\
  \bar{y}_k(0; r) = 0, \bar{y}_k(1; r) = 0.
\end{cases}
\]

Next, the determination of the values of \( c_2(r) \) for all \( r \in [0, 1] \) can be done by solving the nonlinear seventh order HPM series solution of (17) subject to the fuzzy boundary condition \( \bar{y}(1; r) = \frac{5}{4} \left( \frac{r}{10} \right) + \frac{r}{4} \) and \( \bar{y}(1; r) = \frac{5}{4} \left( \frac{r}{10} \right) + \frac{r}{4} \) of (13). Then, we substitute the values of \( c_2(r) \) again in the seventh order series solution of (13) to obtain seventh order HPM series solution in Table 5.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( c_2(r) )</th>
<th>( \bar{y}_0(r) )</th>
<th>( \bar{y}(0.5; r) )</th>
<th>( \bar{y}(0.5; r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.449857550869676</td>
<td>-0.048917383180625</td>
<td>0.001632384089853</td>
<td>0.301632530143259</td>
</tr>
<tr>
<td>0.25</td>
<td>-0.424922206521723</td>
<td>-0.074063710503880</td>
<td>0.020390239397381</td>
<td>0.282982601027203</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.399915734351573</td>
<td>-0.099202436956963</td>
<td>0.039132280515267</td>
<td>0.264132604009526</td>
</tr>
<tr>
<td>0.75</td>
<td>-0.374926306504970</td>
<td>-0.124329636921329</td>
<td>0.057882245488394</td>
<td>0.245382621834460</td>
</tr>
<tr>
<td>1</td>
<td>-0.349923915263898</td>
<td>-0.149445121282975</td>
<td>0.076632205939087</td>
<td>0.226632633151588</td>
</tr>
</tbody>
</table>

5.4. VIM formulation

The VIM to solve (13) is in compliance with section 4. The formulation shall be given as in (18):

\[
\begin{align*}
  y_{i+1} & (t; r; c_2(r)) = y_i (t; r; c_2(r)) \\
& + \int_0^1 \left[ y'' (\eta; r; c_2(r)) - y_i (t; r; c_2(r)) \right] d\eta, \\
  y_{i+1} (t; r; \bar{c}_2(r)) &= \bar{y}_i (t; r; \bar{c}_2(r)) \\
& + \int_0^1 \left[ \bar{y}' (\eta; r; \bar{c}_2(r)) - \bar{y}_i (t; r; \bar{c}_2(r)) \right] d\eta.
\end{align*}
\]

By following VIM formulation and analysis in Example 5.2, third-order VIM series solution is given in the following Table 6:
In order to show efficiency of VIM and HPM methods in solving (13), numerical comparisons of the accuracy generated by RKHS in [35] are presented in Table 7. These comparisons are conducted at \( t = 0.5 \) and various \( r \) fuzzy level that belongs to \([0,1]\) by computing the mean of the average error. Here, \( \tilde{E}(t; r) \) is the mean of the average error between \( \tilde{E}(t; r) \) and \( E(t; r) \) such that

\[
\begin{align*}
\tilde{E}(t; r) &= \frac{1}{N} \sum_{i=1}^{N} (t_i; r_i) - (t_i; r_i), \\
E(t; r) &= \frac{1}{N} \sum_{i=1}^{N} (y_i(t; r_i) - y_i(t; r_i)).
\end{align*}
\]

Table 6. Approximate solution of third order VIM of (13) at \( t = 0.5 \) \( \forall \ r \in [0,1] \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \hat{u}(r) )</th>
<th>( \hat{v}(r) )</th>
<th>( y(0.5; r) )</th>
<th>( \bar{y}(0.5; r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.45000000805284215</td>
<td>-0.050000128774459855</td>
<td>0.076632688439284070</td>
<td>0.3016319259313456</td>
</tr>
<tr>
<td>0.25</td>
<td>-0.4250000216571309</td>
<td>-0.075000108858114996</td>
<td>0.05782686584291370</td>
<td>0.2822820973781950</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.39999981340253</td>
<td>-0.10000090368612918</td>
<td>0.03913267501097500</td>
<td>0.2641321949935540</td>
</tr>
<tr>
<td>0.75</td>
<td>-0.3749999585830666</td>
<td>-0.125000073372107165</td>
<td>0.020382653821232450</td>
<td>0.2453828361308428</td>
</tr>
<tr>
<td>1</td>
<td>-0.3499999563806096</td>
<td>-0.15000057928658794</td>
<td>0.001632623001234793</td>
<td>0.22663236446104182</td>
</tr>
</tbody>
</table>

According to Tables 6 and 7 and Figure 5, we concluded that the third order VIM and seventh order HPM will successfully provide the approximate solutions to (7) for all \( t \in [0,1] \) and \( r \in [0,1] \). This is shown by the results that comply with the fuzzy numbers of properties as triangular fuzzy number.

Figure 5. Exact solution of (13) compared with third order VIM and seventh order HPM approximate solution of (13) at \( t = 0.5 \) and \( r \in [0,1] \)

6. RESULTS COMPARISON

In this section, we present a comparative explanation between the solutions of nonlinear TPFBVPs obtained by HPM and VIM as we illustrate from Examples 5.1 and 5.2:

- The initial approximation guesses in HPM, and VIM are obtained in the same way.
- The construction of VIM formula to solve nonlinear TPFBVP is faster and easier than HPM because HPM takes the advantage of the small parameter \( p \in [0,1] \) that makes HPM suffers from the cumbersome work needed for the derivation of for nonlinear terms. This will increase the computational work especially when the degree of nonlinearity increases.
As mentioned in section 4, VIM is used directly without any requirement or restrictive assumptions that the nonlinear terms make the series solution longer and time consuming in CPU. The third order VIM solved (7) within 31.2264607 seconds for reach r-level values. In HPM the use of embedding parameter p is decomposed for the nonlinear terms making the series solution shorter than the solution of VIM with less time consuming in CPU. The seventh order HPM solved (7) within 1.48B2989 seconds for all r-level values. For illustration, the nonlinear term $y^2$ in (7) which has been decomposed to $\sum_{k=1}^{n-1} y_k y_{n-1-k}$ in HPM formulation but in VIM formula we substitute $y^2$.

From the results obtained by HPM in Tables 1-2 and VIM in Tables 3-4, we conclude that VIM provides a better and more accurate solution than HPM, with less order of series solution for both $t \in [0,1]$ and $r \in [0,1]$. Finally, both Figure 3 and Figure 4 show the solution of (7) by using HPM and VIM respectively for all $t \in [0,1]$ and $r \in [0,1]$ satisfy the fuzzy numbers properties in the form of triangular fuzzy number.

7. CONCLUSION

In this work, approximate analytical methods have been used for nonlinear TPFBVPs to achieve an approximate solution. Two schemes, HPM and VIM, were developed and reformulated to approximate the nonlinear TPFBVP solution. Numerical examples, including nonlinear TPFBVPs, demonstrate the efficacy of these approaches. For nonlinear TPFBVPs, a comparison of HPM and VIM results was presented. The comparison shows that VIM convergence is faster and provides an improved solution, particularly for the less approximate terms, nonlinear TPFBVPs over HPM. Even though these equations are without exact analytical solutions, the exactness of both HPM and VIM can be calculated from nonlinear TPFBVPs. The VIM Lagrangian multiplier for the nonlinear TPFBVPs is equivalent to the value for all the r-level sets. All outcomes of the experiments with HPM and VIM are achieved using a triangular shape to acquire the properties of the fuzzy numbers.

ACKNOWLEDGEMENT

All authors gratefully acknowledge the facilities and financial assistance provided by Universiti Sains Malaysia.

REFERENCES
