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Conformable Chebyshev differential equation of first kind

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ABSTRACT

In this paper, the Chebyshev-I conformable differential equation is considered. A proper power series is examined; there are two solutions, the even solution and the odd solution. The Rodrigues' type formula is also allocated for the conformable Chebyshev-I polynomials.

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1. INTRODUCTION

It is well known that the year 1695 is considered as the birthday of the so called field of fractional calculus. L'Hospital wrote a letter on September 30, 1695 to Leibniz wondering about the notation used in the publications for the derivative of the function

$$f(x) = x, \ \frac{d^n f}{dx^n}.$$

L'Hospital asked Leibniz the following question: what would the result be if $n = \frac{1}{2}$. Leibniz wrote back to L'Hospital: an apparent paradox, from which one day useful consequences will be drawn. Since then, many Mathematicians tried their definitions for a fractional order derivative; most of these definitions are based on integration formulas . The commonly used definitions are those given by Riemann-Liouville, Caputo, and Grünwald-Letnikov of non-integer derivatives. These definitions are summarized in the following lines [1–5]. Let n be a positive integer, $\alpha \in (n-1, n]$, and $x \geq a$.

– Riemann-Liouville (left-sided) α fractional derivative of f is defined by:

$$D_a^\alpha [f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-1-\alpha} f(\xi)\, d\xi,$$

where $\Gamma(x)$ is the well known Gamma function.

– Grünwald-Letnikov (left-sided) α fractional derivative of f is defined by:

$$D_a^\alpha [f(x)] = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{[n]} (-1)^k \frac{\Gamma(\alpha+1) f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}.$$

– Caputo (left-sided) derivative α fractional derivative of f is defined by:

$$D_a^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-1-\alpha} \frac{d^n}{d\xi^n}[f(\xi)]\, d\xi., \quad x \geq a.$$

For a complete list of fractional derivatives, see the survey in [2].

An acceptable definition for fractional derivatives should agree with the ordinary derivatives. As noticed in [3], the only property inherited by all definitions of fractional derivatives is the linearity property; they counted the following drawbacks of one definition or another: the Riemann-Liouville derivative does not reproduce the derivative of a constant to be 0. All definitions do not satisfy the product, quotient, and chain rules. There are other drawbacks for the existing definitions of fractional derivatives.

The well-known Chebyshev differential equation of first kind is given in the form [6–13]:

$$(1-x^2)\, y'' - x\, y' + n^2\, y = 0, \quad n \in N.$$

Expanding the solution using a power series around the ordinary point $x = 0$ yields the well-known Chebyshev polynomials of first kind, $T_n(x)$. For Multivariate case, see [14–16]. The Chebyshev-I polynomials fulfill the following orthogonality conditions [6, 7]:

$$\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx = 0, \ \text{whenever } n \neq m. \tag{1}$$

2. RESEARCH METHOD

In this section, the conformable Chebyshev equation of first kind is defined and solved to give the conformable Chebyshev functions of first kind. For $\alpha \in (0,1]$, the α conformable derivative of y is denoted by $D^{\alpha}y$; the second α conformable derivative is denoted by $D^{\alpha}D^{\alpha}y$. Recently, in [3] the conformable derivative is given in the following definition, see also [4].

Definition Let $f : [0,\infty) \mapsto R$, then the conformable derivative of f of order α is defined by:

$$D^{\alpha}f(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon x^{1-\alpha}) - f(x)}{\epsilon}, \ x > 0, \ \alpha \in (0,1). \tag{2}$$

If f is α differentiable in $(0,a), \ a > 0$, and

$$\lim_{x \to 0^+} D^{\alpha}f(x)$$

exists, then we set

$$D^{\alpha}f(0) = \lim_{x \to 0^+} D^{\alpha}f(x).$$

Let $a,b,c,p \in R$ and f and g be α differentiable at a point $x > 0$, then the conformable derivative satisfies the following properties:

$$\begin{aligned} D^{\alpha}(c) &= 0, \\ D^{\alpha}(x^p) &= px^{p-\alpha}, \\ D^{\alpha}(af+bg) &= aD^{\alpha}f + bD^{\alpha}g. \end{aligned} \tag{3}$$

It is satisfactory to consider the α conformable derivative for $\alpha \in (0,1]$. If $\alpha \in (n, n+1], \ n \in N$, then the conformable derivative, if it exist, is defined by

$$D^{\alpha}f(x) = D^n\left(D^{\alpha-n}f(x)\right).$$

A fractional power series of α is defined as a series of the form

$$\sum_{k=0}^{\infty} a_k x^{k\alpha}, \ \alpha \in (0,1]. \tag{4}$$

Throughout this paper, let $\mathcal{P}_n$be the set of all polynomials of degree $\leq n$.
For $\alpha \in (0,1]$, we define the conformable Chebyshev differential equation of the first kind by:

$$(1-x^{2\alpha})\ D^{\alpha}D^{\alpha}y - \alpha\ x^{\alpha}\ D^{\alpha}y + \alpha^2\ n^2\ y = 0, \quad n \in N. \tag{5}$$

For few applications using the Chebyshev polynomials, see [17–26]. A solution of (5) in a proper power series of α is considered. Let this solution be given by a conformable power series of α as

$$y\ =\ \sum_{k=0}^{\infty} a_k\ x^{k\alpha}.$$

3. RESULTS AND ANALYSIS

Using the rules of conformable derivatives (3), then the α first and second conformable derivatives are given by:

$$D^{\alpha}y = \sum_{k=1}^{\infty} \alpha\ k\ a_k\ x^{(k-1)\alpha},$$

$$D^{\alpha}D^{\alpha}y = \sum_{k=2}^{\infty} \alpha^2\ k(k-1)\ a_k\ x^{(k-2)\alpha}. \tag{6}$$

Substituting the proper power series and its conformable derivatives in (5) yields the following equation:

$$(1-x^{2\alpha})\sum_{k=2}^{\infty}\alpha^2 k(k-1)a_k x^{(k-2)\alpha} - \alpha x^{\alpha}\sum_{k=1}^{\infty}\alpha\ k\ a_k x^{(k-1)\alpha} + \alpha^2\ n^2\ \sum_{k=0}^{\infty} a_k x^{k\alpha} = 0.$$

This is further simplified to

$$\sum_{k=2}^{\infty}\alpha^2 k(k-1)a_k x^{(k-2)\alpha} - \sum_{k=2}^{\infty}\alpha^2 k(k-1)a_k x^{k\alpha} - \alpha x^{\alpha}\sum_{k=1}^{\infty}\alpha k a_k x^{(k-1)\alpha} +$$

$$\alpha^2 n^2\ \sum_{k=0}^{\infty} a_k x^{k\alpha} = 0.$$

In the last equation, substitute $k+2$ for k in the first term to get:

$$\sum_{k=0}^{\infty}\alpha^2(k+2)(k+1)a_{k+2}x^{k\alpha} - \sum_{k=2}^{\infty}\alpha^2 k(k-1)a_k x^{k\alpha} - \sum_{k=1}^{\infty}\alpha^2 k a_k x^{k\alpha} +$$

$$\sum_{k=0}^{\infty}\alpha^2 n^2 a_k x^{k\alpha} = 0.$$

Rewriting the last equation so that the summations start counting with $k = 2$ yields:

$$(2\alpha^2 a_2 + \alpha^2 n^2 a_0) + \left[6\alpha^2 a_3 - \alpha^2 a_1 + \alpha^2 n^2 a_1\right]\ x^{\alpha}$$

$$+\sum_{k=2}^{\infty}\left[\alpha^2(k+2)(k+1)a_{k+2} - \alpha^2 k(k-1)a_k - \alpha^2 k a_k + \alpha^2 n^2 a_k\right]\ x^{k\alpha} = 0.$$

To compute the values of the parameters of the fractional power series, the coefficients on both sides are compared. The constant coefficient yields

$$2\alpha^2 a_2 + \alpha^2 n^2 a_0 = 0.$$

Solving for a_2 gives

$$a_2 = \frac{-n^2}{2}\, a_0. \tag{7}$$

The coefficient of x^α yields

$$6\alpha^2 a_3 - \alpha^2 a_1 + \alpha^2 n^2 a_1 = 0.$$

Solving for a_3 gives

$$a_3 = -\frac{n^2-1}{6}\, a_1. \tag{8}$$

The general term for the coefficient of $x^{k\alpha}$ yields

$$\alpha^2(k+2)(k+1)a_{k+2} - \alpha^2 k(k-1)a_k - \alpha^2 k a_k + \alpha^2 n^2 a_k = 0.$$

This is solved for a_{k+2} to yield:

$$a_{k+2} = \frac{k^2-n^2}{(k+2)(k+1)}\, a_k.$$

This can be rewritten in the form

$$a_{k+2} = -\frac{(n-k)(n+k)}{(k+2)(k+1)}\, a_k. \tag{9}$$

The fractional power series contains either even or odd terms. Consequently, there are two independent solutions. The first solution is the even-terms solution, and the second solution is the odd-terms solution. Both solutions diverge at $x = \pm 1$. The only interesting solutions are polynomial solutions. For the even terms, substituting $k = 0$ in the last formula gives (7) for a_2. Substituting $k = 2, 4, 6, \ldots$ in the last formula gives $a_4, a_6, a_8, \ldots$ as follows:

$$a_4 = \frac{(n-2)(n+2)}{4(3)} \frac{n^2}{2}\, a_0,$$

$$a_6 = -\frac{(n-4)(n+4)}{6(5)} \frac{(n-2)(n+2)}{4(3)} \frac{n^2}{2}\, a_0,$$

$$a_8 = \frac{(n-6)(n+6)}{8(7)} \frac{(n-4)(n+4)}{6(5)} \frac{(n-2)(n+2)}{4(3)} \frac{n^2}{2}\, a_0.$$

The general term looks as follows:

$$a_{2k} = (-1)^k \frac{(n-2(k-1))(n+2(k-1))}{2k(2k-1)} \cdots \frac{(n-4)(n+4)}{6(5)} \frac{(n-2)(n+2)}{4(3)} \frac{n(n)}{2(1)}\, a_0.$$

This can be written as follows:

$$a_{2k} = \frac{(-1)^k}{(2k)!} \prod_{i=0}^{k-1} (n-2i)(n+2i)\, a_0, \tag{10}$$

where $\prod$ is the product symbol. If we set

$$a_0 = (-1)^{[\frac{n}{2}]}$$

gives the following conformable Chebyshev polynomials:

$$T_0(x) = 1,$$

$$T_2(x) = 2x^{2\alpha} - 1,$$

$$T_4(x) = 8x^{4\alpha} - 8x^{2\alpha} + 1,$$
$$T_6(x) = 32x^{6\alpha} - 48x^{4\alpha} + 18x^{2\alpha} - 1,$$
$$T_8(x) = 128x^{8\alpha} - 256x^{6\alpha} + 160x^{4\alpha} - 32x^{2\alpha} + 1.$$

Similarly, when the fractional power series contains only odd terms, then the odd terms are given in the following compact form:

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!} \prod_{i=0}^{k-1} (n-1-2i)(n+1+2i)\, a_1. \tag{11}$$

If we set

$$a_1 = (-1)^{[\frac{n}{2}]} n$$

gives the following conformable Chebyshev polynomials:

$$T_1(x) = x^{\alpha},$$
$$T_3(x) = 4x^{3\alpha} - 3x^{\alpha},$$
$$T_5(x) = 16x^{5\alpha} - 20x^{3\alpha} + 5x^{\alpha},$$
$$T_7(x) = 64x^{7\alpha} - 112x^{5\alpha} + 56x^{3\alpha} - 7x^{\alpha}.$$

To derive a Rodrigues' type formula for the Chebyshev-I polynomials, $T_n(x)$, a polynomial $t_n(x)$ is to be necessitated that fulfills the Chebyshev orthogonality:

$$\int_{-1}^{1} \frac{t_n(x) p_{n-1}(x)}{\sqrt{1-x^2}}\, dx = 0,\ \forall p_{n-1} \in \mathcal{P}_{n-1}. \tag{12}$$

Rewrite the integrand in (12) in the form

$$\frac{t_n(x)}{\sqrt{1-x^2}}\, p_{n-1}(x) = \left(\frac{d^n}{dx^n} \phi_n(x) \right) p_{n-1}(x)$$

and integrate the orthogonality conditions by parts n times to get

$$0 \ = \ \int_{-1}^{1} \left(\frac{d^n}{dx^n} \phi_n(x) \right) p_{n-1}(x)\, dx = \sum_{i=0}^{n-1} (-1)^i \phi_n^{(n-1-i)}(x)\, p_{n-1}^{(i)}(x) \Bigg]_{x=-1}^{1}. \tag{13}$$

The last formula is valid for any polynomial of degree $\leq n-1$, thus we acquire the following conditions on $\phi_n(x)$ and its derivatives at $x = -1, 1$ as follows

$$\begin{aligned} \phi_n(-1) = \phi_n'(-1) = \cdots = \phi_n^{(n-1)}(-1) = 0, \\ \phi_n(1) = \phi_n'(1) = \cdots = \phi_n^{(n-1)}(1) = 0. \end{aligned} \tag{14}$$

The polynomial $t_n(x)$ is of degree n and thus

$$0 \ = \ \frac{d^{n+1} t_n(x)}{dx^{n+1}} = \frac{d^{n+1}}{dx^{n+1}} \left(\sqrt{1-x^2}\, \frac{d^n \phi_n(x)}{dx^n} \right). \tag{15}$$

The (15) with the conditions in (14) form a differential equation with $2n$ boundary conditions and has the solution

$$\phi_n(x) = b_n \left(1-x^2\right)^{n-\frac{1}{2}},\ \text{ for some constant } b_n. \tag{16}$$

Consequently, the polynomial $t_n(x)$ has the form

$$t_n(x) \;=\; b_n\,\sqrt{1-x^2}\,\frac{d^n}{dx^n}\left[\left(1-x^2\,\right)^{n-\frac{1}{2}}\right]. \tag{17}$$

Normalizing

$$t_n(1) \;=\; T_n(1)$$

yields

$$(-1)^n(2n-1)(2n-3)\cdots(5)(3)(1)\,b_n = 1.$$

And thus

$$b_n \;=\; \frac{(-1)^n}{(2n-1)!!}, \tag{18}$$

where $(2n-1)!! = (2n-1)(2n-3)(\cdots(3)(1)$ denotes the double factorial. Since

$$(2n-1)!! = \frac{(2n)!}{n!2^n}$$

thus

$$b_n \;=\frac{(-2)^n\;n!}{(2n)!}. \tag{19}$$

Thus, the Chebyshev-I polynomials are given by the following Rodrigues' type formula:

$$T_n(x) = \frac{(-2)^n n!}{(2n)!}\,(1-x^2)^{\frac{1}{2}}\,\frac{d^n}{dx^n}\left[(1-x^2)^{n-\frac{1}{2}}\right]. \tag{20}$$

The conformable Chebyshev-I polynomials can be expressed by the following Rodrigues' type formula:

$$T_n(x) = \frac{(-2)^n n!}{(2n)!\alpha^n}\,(1-x^{2\alpha})^{\frac{1}{2}}\,D^{n\alpha}\left[(1-x^{2\alpha})^{n-\frac{1}{2}}\right],\ n\in N,\ \alpha\in(0,1]. \tag{21}$$

It is also interesting to discuss and find a similar generalization to orthogonal polynomials on triangular domains [14].

4. CONCLUSION

In this paper, the conformable Chebyshev differential equation of the first kind is introduced in formula (5), using the conformable derivative defined in formula (2). Then the explicit form of the conformable Chebyshev functions are derived in the form of a power series and a Ridrigue's type formula is also derived in (20). We consider the conformable derivative given in formula (2) of a function $f(x)$. This means that the ordinary derivative $f'(x)$ exists and $f'(x) = D^1 f(x)$. Moreover, $D^\alpha f(x) = x^{1-\alpha}f'(x)$ and $D^\alpha D^\alpha f(x) = (1-\alpha)x^{1-2\alpha}f'(x)+x^{2-2\alpha}f''(x)$ for $\alpha\in(0,1]$. From these formulas, it is clear that, if the ordinary Chebyshev polynomials $y(x) = T_n(x)$ are solutions of the ordinary Chebyshev differential equation, then the functions $u(x) = y(x^\alpha) = T_n(x^\alpha)$ are solutions of the conformable differential (5).

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Abedallah Rababah is a professor of mathematics at United Arab Emirates University and is on leave from Jordan University of Science and Technology. He is working in the field of Computer Aided Geometric Design, abbreviated CAGD. In particular, his research is on degree raising and reduction of Bézier curves and surfaces with geometric boundary conditions, Bernstein polynomials, and their duality. He is known for his research in describing approximation methods that significantly improve the standard rates obtained by classical (local Taylor, Hermite) methods. He proved the following conjecture for a particular set of curves of nonzero measure: Conjecture: A smooth regular planar curve can, in general, be approximated by a polynomial curve of degree n with order 2n. The method exploited the freedom in the choice of the parametrization and achieved the order 4n/3, rather than n + 1. Generalizations were also proved for space curves. Professor Rababah is also doing research in the fields of classical approximation theory, orthogonal polynomials, Jacobi-weighted orthogonal polynomials on triangular domains, and best uniform approximations. Since 1992, He has been teaching at German, Jordanian, American, Canadian, and Emirates' universities. He is active in the editorial boards of many journals in mathematics and computer science. Further info can be found on his homepage at ResearchGate:
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