$\epsilon_\varphi$-contraction and some fixed point results via modified $\omega$-distance mappings in the frame of complete quasi metric spaces and applications

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ABSTRACT

In this Article, we introduce the notion of an $\epsilon_\varphi$-contraction, which is based on modified $\omega$-distance mappings, and employ this new definition to prove some fixed point result. Moreover, to highlight the significance of our work, we present an interesting example along with an application.

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1. INTRODUCTION

One of the most important methods in mathematics used to discuss the existence and uniqueness of a solution of such equations is the Banach contraction principle [1]. It is considered as a valuable tool in fixed point theory. Since then, many mathematicians investigated the Banach contraction principle in many directions. In [2], Abodayeh et al. utilized the concept of $\Omega$–distance to give some new generalizations of Banach contraction principle. Shatanawi, M. Postolache in [3, 4] studied some common fixed points of such mappings. For more generalizations of Banach fixed point theory, see [5–18]. In 1931 Wilson [19] introduced the notion of quasi metric space as below:

**Definition 1** [19] We call the function $q : E \times E \to [0, \infty)$ a quasi metric if it satisfies:

(i) $q(e_1, e_2) = 0 \Leftrightarrow e_1 = e_2$;

(ii) $q(e_1, e_2) \leq q(e_1, e_3) + q(e_3, e_1)$ for all $e_1, e_2, e_3 \in E$.

The pair $(E, q)$ is called a quasi metric space.

For some work in quasi metric spaces, see [20–23]

If the symmetry condition is added to $(E, q)$ (i.e. $q(e_1, e_2) = q(e_2, e_1)$ for all $e_1, e_2 \in E$), then the space $(E, q)$ is a metric space.

Henceforth, we denote by $(E, q)$ a quasi metric space. To generate a metric $d$ on $E$. Define $d : E \times E \to [0, \infty)$ by

$$d = \max\{q(e_1, e_2), q(e_2, e_1)\}.$$

The concepts of completeness and convergence of quasi metric spaces are given below:

**Definition 2** [24, 25] A sequence $(e_n)$ converges to $e^* \in E$ if $\lim_{s \to \infty} q(e_s, e^*) = \lim_{s \to \infty} q(e^*, e_s) = 0$.

**Definition 3** [25] Let $(e_n)$ be a sequence in $E$. Then we call

(i) $(e_n)$ left-Cauchy if for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that $q(e_s, e_t) < \delta$ for all $s \geq t > N_0$.

(ii) $(e_n)$ right-Cauchy if for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that $q(e_t, e_s) < \delta$ for all $t \geq s > N_0$.

**Definition 4** [24, 25] A sequence $(e_n)$ in $E$ is called a Cauchy sequence if

(i) If for any $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that $q(e_s, e_t) \leq \delta$ for all $s, t > N_0$;

or

(ii) $(e_n)$ is right and left Cauchy.

**Definition 5** [24, 25] We say $(E, q)$ is complete if every Cauchy sequence $(e_n)$ in $E$ is convergent.

In 2016, Alegre and Marin [26] introduced the notion of modified $\omega$-distance mappings on $(E, q)$.

**Definition 6** [26] A modified $\omega$-distance (shortly m$\omega$-distance ) on $(E, q)$ is a function $\rho : E \times E \to [0, \infty)$, which satisfies the following:

(W1) $\rho(e_1, e_2) \leq \rho(e_1, e_3) + \rho(e_3, e_2)$ for all $e_1, e_2, e_3 \in E$;

(W2) $\rho(e, .) : E \to [0, \infty)$ is lower semi-continuous for all $e \in E$; and

(mW3) for each $\varrho > 0$ there exists $\delta > 0$ such that if $\rho(e_1, e_2) \leq \delta$ and $\rho(e_2, e_3) \leq \delta$, then $q(e_1, e_3) \leq \varrho$ for all $e_1, e_2, e_3 \in E$.

Henceforth, we denote by $\rho$ an m$\omega$-distance mapping.

**Definition 7** [26] If $\rho$ is lower semi-continuous on the first and second coordinates, then $\rho$ is called a strong m$\omega$-distance.

**Remark 1** [26] Every quasi metric $q$ on $E$ is m$\omega$-distance.

**Lemma 1** [33] Let $(\varrho_n)$, $(\sigma_n)$ be two sequences of nonnegative real numbers that converge to zero. Then we have the following:

(i) If $\rho(e_s, e_t) \leq \varrho_n$ for all $s, t \in \mathbb{N}$ with $t \geq s$, then $(e_n)$ is right Cauchy in $(E, q)$.

(ii) If $\rho(e_s, e_t) \leq \sigma_n$ for all $s, t \in \mathbb{N}$ with $t \leq s$, then $(e_n)$ is left Cauchy in $(E, q)$.

**Remark 2** [33] The above lemma show that if $\lim_{s, t \to \infty} \rho(e_s, e_t) = 0$, then $(e_n)$ is Cauchy in $(E, q)$.

For more results in fixed point theory in $\omega$ and modified $\omega-$distances, we ask the readers to consider [20, 27–31, 33, 34].

**Definition 8** [35] A self function $\varphi$ on $[0, \infty)$ is said to be an ultra distance function if $\varphi$ satisfies $\varphi(\mu^*) = 0$ $\iff \mu^* = 0$ and if $(\mu^*_s)$ is a sequence in $[0, \infty)$ such that $\lim_{s \to +\infty} \varphi(\mu^*_s) = 0$, then $\lim_{s \to +\infty} \mu^*_s = 0$. 

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2. MAIN RESULTS

The definition of $\epsilon_{\varphi}$-contraction on a pair of self mappings is defined as follows:

**Definition 9** Equipped $(E, q)$ with $\rho$ and let $F, T$ be two self mappings on $E$. Then the pair $(F, T)$ is called an $\epsilon_{\varphi}$-contraction if there exists an ultra distance function $\varphi$ and a given $\epsilon > 0$ such that for all $e_1, e_2 \in E$ we have:

$$\varphi(Fe_1, Te_2) \leq \frac{\rho(e_1, e_2)}{\epsilon + \rho(e_1, Fe_1)} \max \left\{ \varphi(e_1, F e_1), \varphi(e_2, T e_2) \right\}.$$  

And $$\varphi(Te_1, Fe_2) \leq \frac{\rho(e_1, e_2)}{\epsilon + \rho(e_1, Te_1)} \max \left\{ \varphi(e_1, T e_1), \varphi(e_2, F e_2) \right\}.$$ 

Next, we introduce our first result:

**Theorem 2** Equipped $(E, q)$ with $\rho$ and let $F, T$ be two self mappings on $E$ such that the pair $(F, T)$ is an $\epsilon_{\varphi}$-contraction. Also, assume $\rho(e_j, e_{j+1}) = 0$ or $\rho(e_j, e_{j+1}) = 0$, for some $j \in \mathbb{N} \cup \{0\}$. Then $e_j$ is a unique common fixed point of $F$ and $T$ in $E$.

**Proof.** Let $e_0 \in E$. We create a sequence $(e_j)$ in $E$ inductively by taking $Fe_2j = e_{2j+1}$ and $Te_{2j+1} = e_{2j+2}$ for all $j \in \mathbb{N} \cup \{0\}$.

To prove the result, we have to consider the following cases:

**Case(1):** $\rho(e_j, e_{j+1}) = 0$. If $j$ is even, then $j = 2k$ for some $k \in \mathbb{N} \cup \{0\}$, so we have $\rho(e_{2k}, e_{2k+1}) = 0$ and so $\varphi(e_{2k}, e_{2k+1}) = 0$.

Now, since the pair $(F, T)$ is an $\epsilon_{\varphi}$-contraction, we get:

$$\varphi(e_{2k+1}, e_{2k+2}) = \varphi(Fe_{2k}, Te_{2k+1})$$

$$\leq \frac{\rho(e_{2k+1}, e_{2k+2})}{\epsilon + \rho(e_{2k+1}, Fe_{2k})} \max \left\{ \varphi(e_{2k}, Fe_{2k}), \varphi(e_{2k+1}, T e_{2k+1}) \right\}$$

$$= \frac{\rho(e_{2k+1}, e_{2k+2})}{\epsilon + \rho(e_{2k+1}, e_{2k+1})} \max \left\{ \varphi(e_{2k}, e_{2k+1}), \varphi(e_{2k+1}, e_{2k+2}) \right\}$$

$$= 0.$$

By the definition of $\varphi$, we have

$$\rho(e_{2k+1}, e_{2k+2}) = 0.$$  

(1)

From the assumption we have $\rho(e_{2k}, e_{2k+1}) = 0$ and by (1) we get that

$$\rho(e_{2k}, e_{2k+2}) = 0.$$  

(2)

Also, by using mW3 of the definition of $\rho$, we get that

$$q(e_{2k}, e_{2k+2}) = 0.$$  

(3)

$$\varphi(e_{2k+2}, e_{2k+1}) = \varphi(Te_{2k+1}, Fe_{2k})$$

$$\leq \frac{\rho(e_{2k+1}, e_{2k+2})}{\epsilon + \rho(e_{2k+1}, T e_{2k+1})} \max \left\{ \varphi(e_{2k+1}, T e_{2k+1}), \varphi(e_{2k+2}, T e_{2k+2}) \right\}$$

$$= \frac{\rho(e_{2k+1}, e_{2k+2})}{\epsilon + \rho(e_{2k+1}, e_{2k+2})} \max \left\{ \varphi(e_{2k+1}, e_{2k+1}), \varphi(e_{2k+2}, e_{2k+2}) \right\}$$

$$= 0.$$

Therefore,

$$\rho(e_{2k+2}, e_{2k+1}) = 0.$$  

(4)

$\epsilon_{\varphi}$-contraction and some fixed point results ... (K. Abodayeh)
Also, using the Equations (2), (4) and mW3 of the definition of $\rho$, we get that

$$q(e_{2k+1}, e_{2k}) = 0. \quad (5)$$

Hence, $e_{2k} = e_{2k+1} = e_{2k+2}$ and so $e_j$ is a common fixed point of $F$ and $T$ in $E$.

If $j$ is odd, then $j = 2k+1$, for some $k \in \mathbb{N} \cup \{0\}$. Then we have $\rho(e_{2k+1}, e_{2k+2}) = 0$ and hence $\varphi(\rho(e_{2k+1}, e_{2k+2})) = 0$.

Thus, $\varphi(\rho(e_{2k+2}, e_{2k+3})) = \varphi(T(e_{2k+1}, F(e_{2k+2}))$.

For all $j$, we can prove that $\varphi(\rho(e_{2k+1}, e_{2k+2})) = 0$.

Also, using the Equations (2), (4) and mW3 of the definition of $\rho$, we get that

$$\varphi(\rho(e_{2k+2}, e_{2k+3})) = \varphi(T(e_{2k+1}, F(e_{2k+2}))$.

Let $L = \frac{\rho(e_{2k+1}, e_{2k+2})}{\epsilon + \rho(e_{2k+1}, e_{2k+2})}$. Then $L < 1$ and so

$$\varphi(\rho(e_{2k+2}, e_{2k+3})) < \varphi(\rho(e_{2k+2}, e_{2k+3}))$$

Thus, $\varphi(\rho(e_{2k+2}, e_{2k+3})) = 0$. By the definition $\varphi$, we get that

$$\rho(e_{2k+2}, e_{2k+3}) = 0. \quad (6)$$

From the assumption, we have $\rho(e_{2k+1}, e_{2k+2}) = 0$ and by (6), we get

$$\rho(e_{2k+1}, e_{2k+3}) = 0. \quad (7)$$

Also, Condition mW3 of the definition of $\rho$ implies that

$$q(e_{2k+1}, e_{2k+3}) = 0. \quad (8)$$

In a similar manner, we can prove that if $\rho(e_{j+1}, e_j) = 0$, then $e_j$ is a common fixed point of $F$ and $T$ in $E$. \qed

Next, we introduce our main result:

**Theorem 3** Equipped $(E, q)$ with $\rho$ and let $F, T$ be two self mappings on $E$. Assume the following conditions hold:

(i) $(E, q)$ is complete;

(ii) The pair $(F, T)$ is an $\epsilon_\rho$-contraction;

(iii) $F$ and $T$ are continuous;

(iv) For all $e_1, e_2 \in E$ and some integer $I$ we have $\rho(e_1, e_2) \leq L$.

Then $F$ and $T$ have a unique common fixed point in $E$.

**Proof.** Let $e_0 \in E$. Construct a sequence $(e_n)$ in $E$ inductively by taking $Fe_{2n} = e_{2n+1}$ and $Te_{2n+1} = e_{2n+2}$ for all $n \in \mathbb{N} \cup \{0\}$.

If for some $i \in \mathbb{N}$ we have $\rho(e_i, e_{i+1}) = 0$ or $\rho(e_{i+1}, e_i) = 0$, then by Theorem 2, $e_i$ is a unique common fixed point of $F$ and $T$ in $E$. 

Now, assume that $\rho(e_n, e_{n+1}) \neq 0$ and $\rho(e_{n+1}, e_n) \neq 0$, for all $n \in \mathbb{N} \cup \{0\}$. Since the pair $(F, T)$ is an $\epsilon_\varphi$-contraction, then we have

$$\varphi \rho(e_{2n+1}, e_{2n+2}) = \varphi \rho(Fe_{2n}, Te_{2n+1}) \leq \left(\frac{\rho(e_{2n}, e_{2n+1})}{\epsilon + \rho(e_{2n}, e_{2n+1})}\right) \max \left\{ \varphi \rho(e_{2n}, Fe_{2n}), \varphi \rho(e_{2n+1}, Te_{2n+1}) \right\} = \left(\frac{\rho(e_{2n}, e_{2n+1})}{\epsilon + \rho(e_{2n}, e_{2n+1})}\right) \max \left\{ \varphi \rho(e_{2n}, e_{2n+1}), \varphi \rho(e_{2n+1}, e_{2n+2}) \right\}.$$  

If $L = \frac{\rho(e_{2n}, e_{2n+1})}{\epsilon + \rho(e_{2n}, e_{2n+1})}$, then $L < 1$.

Also, if $\max \left\{ \varphi \rho(e_{2n}, e_{2n+1}), \varphi \rho(e_{2n+1}, e_{2n+2}) \right\} = \varphi \rho(e_{2n+1}, e_{2n+2})$, we get that

$$\varphi \rho(e_{2n+1}, e_{2n+2}) \leq L \max \left\{ \varphi \rho(e_{2n}, e_{2n+1}), \varphi \rho(e_{2n+1}, e_{2n+2}) \right\} = L \varphi \rho(e_{2n+1}, e_{2n+2}) < \varphi \rho(e_{2n+1}, e_{2n+2}).$$

Thus, $\varphi \rho(e_{2n+1}, e_{2n+2}) = 0$ and so $\rho(e_{2n+1}, e_{2n+2}) = 0$ a contradiction. Therefore,

$$\varphi \rho(e_{2n+1}, e_{2n+2}) \leq \left(\frac{\rho(e_{2n}, e_{2n+1})}{\epsilon + \rho(e_{2n}, e_{2n+1})}\right) \varphi \rho(e_{2n}, e_{2n+1}).$$  

$$\varphi \rho(e_{2n+2}, e_{2n+1}) = \varphi \rho(Te_{2n+1}, Fe_{2n}) \leq \left(\frac{\rho(e_{2n+1}, e_{2n+2})}{\epsilon + \rho(e_{2n+1}, e_{2n+2})}\right) \max \left\{ \varphi \rho(e_{2n+1}, e_{2n+2}), \varphi \rho(e_{2n}, e_{2n+1}) \right\} = \left(\frac{\rho(e_{2n+1}, e_{2n+2})}{\epsilon + \rho(e_{2n+1}, e_{2n+2})}\right) \max \left\{ \varphi \rho(e_{2n+1}, e_{2n+2}), \varphi \rho(e_{2n}, e_{2n+1}) \right\} = \left(\frac{\rho(e_{2n+1}, e_{2n+2})}{\epsilon + \rho(e_{2n+1}, e_{2n+2})}\right) \varphi \rho(e_{2n}, e_{2n+1}).$$

Also, we can show that:

$$\varphi \rho(e_n, e_{n+1}) \leq \left(\frac{\rho(e_{n-1}, e_n)}{\epsilon + \rho(e_{n-1}, e_n)}\right) \varphi \rho(e_{n-1}, e_n).$$  

And

$$\varphi \rho(e_{n+1}, e_n) \leq \left(\frac{\rho(e_n, e_{n-1})}{\epsilon + \rho(e_n, e_{n-1})}\right) \varphi \rho(e_{n-1}, e_n).$$

Now,

$$\varphi \rho(e_n, e_{n+1}) \leq \left(\frac{\rho(e_{n-1}, e_n)}{\epsilon + \rho(e_{n-1}, e_n)}\right) \varphi \rho(e_{n-1}, e_n) \leq \left(\frac{\rho(e_{n-1}, e_n)}{\epsilon + \rho(e_{n-1}, e_n)}\right) \left(\frac{\rho(e_{n-2}, e_{n-1})}{\epsilon + \rho(e_{n-2}, e_{n-1})}\right) \varphi \rho(e_{n-2}, e_{n-1}) \leq \cdots \leq \prod_{i=1}^{n} \left(\frac{\rho(e_{n-i}, e_i)}{\epsilon + \rho(e_{n-i}, e_i)}\right) \varphi \rho(e_0, e_1).$$

Let $L_i = \left(\frac{\rho(e_{n-i}, e_i)}{\epsilon + \rho(e_{n-i}, e_i)}\right)$. Then $L_i < 1$ for all $i \in \{1, 2, \cdots, n\}$, so we have

$$\varphi \rho(e_n, e_{n+1}) \leq \prod_{i=1}^{n-1} L_i(\varphi \rho(e_n, e_{n+1})).$$  

$\epsilon_\varphi$-contraction and some fixed point results ... (K. Abodayeh)
Letting $n \to \infty$, we get
\[
\lim_{n \to \infty} \varphi p(e_n, e_{n+1}) = 0. \tag{14}
\]

Since $\varphi$ is ultra distance function, we have
\[
\lim_{n \to \infty} \rho(e_n, e_{n+1}) = 0. \tag{15}
\]
\[
\varphi p(e_{n+1}, e_n) \leq \left( \frac{\rho(e_n, e_{n+1})}{\rho(e_n, e_{n+1})} \right) \varphi p(e_{n-1}, e_n)
\leq \left( \frac{\rho(e_n, e_{n+1})}{\rho(e_n, e_{n+1})} \right) \left( \frac{\rho(e_{n-2}, e_{n-1})}{\rho(e_{n-2}, e_{n-1})} \right) \varphi p(e_{n-2}, e_{n-1})
\leq \cdots \leq \left( \frac{\rho(e_n, e_{n+1})}{\rho(e_n, e_{n+1})} \right) \left( \frac{\rho(e_{n-1}, e_1)}{\rho(e_{n-1}, e_1)} \right) \varphi p(e_0, e_1).
\]

Let $L_i = \left( \frac{\rho(e_{i-1}, e_i)}{\rho(e_{i-1}, e_i)} \right)$. Then $L_i < 1$ for all $i \in \{1, 2, \cdots, n-1\}$ and since $\rho(e_1, e_2) \leq L$ for all $e_1, e_2 \in E$ and some integer $L$, we get that
\[
\varphi p(e_{n+1}, e_n) \leq L \prod_{i=1}^{n-1} L_i(\varphi p(e_0, e_1)). \tag{16}
\]

Letting $n \to \infty$, we get that:
\[
\lim_{n \to \infty} \varphi p(e_{n+1}, e_n) = 0. \tag{17}
\]

The definition of $\varphi$ informs us
\[
\lim_{n \to \infty} \rho(e_{n+1}, e_n) = 0. \tag{18}
\]

Now, we need to show that $(e_s)$ is a Cauchy sequence in $E$.

In order to do that, we first prove that $(e_s)$ is a right Cauchy sequence in $(E, q)$. For each $s, t \in \mathbb{N}$ with $s < t$, we have the following cases:

**Case (1):** If $s$ odd and $t$ even, then we have:
\[
\varphi p(e_s, e_t) = \varphi(F e_{s-1}, T e_{t-1})
\leq \left( \frac{\rho(e_{s-1}, e_1)}{\rho(e_{s-1}, e_1)} \right) \max \left\{ \varphi p(e_{s-1}, F e_{s-1}), \varphi p(e_{t-1}, T e_{t-1}) \right\}
\leq \left( \frac{\rho(e_{s-1}, e_1)}{\rho(e_{s-1}, e_1)} \right) \max \left\{ \varphi p(e_{s-1}, e_s), \varphi p(e_{t-1}, e_t) \right\},
\]
\[
= \left( \frac{\rho(e_{s-1}, e_1)}{\rho(e_{s-1}, e_1)} \right) \varphi p(e_{s-1}, e_s).
\]

Let $L_i = \left( \frac{\rho(e_{i-1}, e_i)}{\rho(e_{i-1}, e_i)} \right)$. Since $\rho(e_1, e_2) \leq L$ for all $e_1, e_2 \in E$ and some integer $L$, we have
\[
\varphi p(e_s, e_t) \leq L \prod_{i=1}^{s-1} L_i(\varphi p(e_0, e_1)). \tag{19}
\]

Letting $s, t \to \infty$, we have $\lim_{s, t \to \infty} \varphi p(e_s, e_t) = 0$.

Thus,
\[
\lim_{s, t \to \infty} \varphi p(e_s, e_t) = 0. \tag{20}
\]
In a similar manner, we can prove that

To prove the uniqueness of $\rho(e_{-1}, Fe_{-1})$

Using Lemma 1, we get that

If $s$ is a Cauchy sequence in $E$, such that

Letting $s,t \to \infty$, we have $\lim_{s,t \to \infty} \rho(e_{s+1}, e_t) = 0$.

Case (2): If $s$ even and $t$ odd, then we have:

$$
\varphi \rho(e_s, e_t) = \varphi \rho(Te_{s-1}, Fe_{t-1}) \\
\leq \max \left\{ \varphi \rho(e_{s-1}, Te_{s-1}), \varphi \rho(e_{t-1}, Fe_{t-1}) \right\} \\
= \varphi \rho(e_{s-1}, e_s) \varphi \rho(e_{t-1}, e_t) \\
= \varphi \rho(e_{s-1}, e_s).
$$

Let $L_i = \left( \frac{\rho(e_{i-1}, e_i)}{\varphi \rho(e_{i-1}, e_i)} \right)$. Since $\rho(e_{i-1}, e_i) \leq L$ for all $e_{i-1}, e_i \in E$ and some integer $L$, then we get that

$$
\varphi \rho(e_s, e_t) \leq L \prod_{i=1}^{s-1} L_i(\varphi \rho(e_{i}, e_{i})).
$$

(21)

Letting $s,t \to \infty$, we have $\lim_{s,t \to \infty} \varphi \rho(e_s, e_t) = 0$.

So,

$$
\lim_{s,t \to \infty} \varphi \rho(e_s, e_t) = 0.
$$

(22)

Case (3): If $s$ and $t$ are odd, we get

$$
\rho(e_s, e_t) \leq \rho(e_s, e_{s+1}) + \rho(e_{s+1}, e_t).
$$

(23)

Hence,

$$
\lim_{s,t \to \infty} \rho(e_s, e_t) = 0.
$$

(24)

Case (4): If $s$ and $t$ are even, we get

$$
\rho(e_s, e_t) \leq \rho(e_s, e_{t-1}) + \rho(e_{t-1}, e_t).
$$

(25)

Hence,

$$
\lim_{s,t \to \infty} \rho(e_s, e_t) = 0.
$$

(26)

Using Lemma 1, we get that $(e_s)$ is a right Cauchy sequence in $(E,q)$. Similarly, we can prove that $(e_s)$ is a left Cauchy sequence in $E$.

Hence, $(e_s)$ is a Cauchy sequence in $E$. The completeness of $(E,q)$ implies that there exists an element $e^* \in E$ such that $(e_s) \to e^*$.

If $F$ is a continuous function then $e_{s+1} = Fe_s \to Fe^*$. By the uniqueness of limit, we get that $Fe^* = e^*$. In a similar manner, we can prove that $Te^* = e^*$ when $T$ is a continuous function.

To prove the uniqueness of $e^*$. First we show that $\rho(e^*, e^*) = 0$.

$$
\varphi \rho(e^*, e^*) = \varphi \rho(Fe^*, Fe^*) \\
\leq \max \left\{ \varphi \rho(e^*, Fe^*), \varphi \rho(e^*, e^*) \right\} \\
= \max \left\{ \varphi \rho(e^*, e^*), \varphi \rho(e^*, e^*) \right\} \\
= 0.
$$

Therefore, $\rho(e^*, e^*) = 0$. 

$\epsilon$-contraction and some fixed point results ... (K. Abodayeh)
Assume that there exists $\mu^* \in E$ such that $F\mu^* = T\mu^* = \mu^*$. Then
\[
\varphi \rho(e^*, \mu^*) = \varphi \rho(Fe^*, T\mu^*) \\
\leq \left( \frac{\rho(e^*, \mu^*)}{\epsilon + \rho(e^*, Fe^*)} \right) \max \left\{ \varphi \rho(e^*, Fe^*), \varphi \rho(\mu^*, T\mu^*) \right\} \\
= \left( \frac{\rho(e^*, \mu^*)}{\epsilon + \rho(e^*, e^*)} \right) \max \left\{ \varphi \rho(e^*, e^*), \varphi \rho(\mu^*, \mu^*) \right\} \\
= 0.
\]

Thus, we have $\rho(e^*, \mu^*) = 0$ since $\rho(e^*, e^*) = 0$ we get that $q(e^*, \mu^*) = 0$ and so $e^* = \mu^*$. □

**Corollary 4** A complete $(E, q)$ equipped with $\rho$ and let $F, T$ be two self continuous mappings on $E$. Assume the following conditions hold:

(i) For all $e_1, e_2 \in E$ and a given $\epsilon > 0$ and an ultra distance function $\varphi$ we have:

\[
\varphi \rho(Fe_1, Te_2) \leq \left( \frac{\rho(e_1, e_2)}{2(\epsilon + \rho(e_1, Ve_1))} \right) \left( \varphi \rho(e_1, Fe_1) + \varphi \rho(e_2, Te_2) \right).
\]

And

\[
\varphi \rho(Te_1, Fe_2) \leq \left( \frac{\rho(e_1, e_2)}{2(\epsilon + \rho(e_1, Te_1))} \right) \left( \varphi \rho(e_1, Te_1) + \varphi \rho(e_2, Fe_2) \right).
\]

(ii) For all $e_1, e_2 \in E$ we have $\rho(e_1, e_2) \leq L$ for some integer $L$.

Then $F$ and $T$ have a unique common fixed point in $E$.

**Proof.**
\[
\varphi \rho(Fe_1, Te_2) \leq \left( \frac{\rho(e_1, e_2)}{2(\epsilon + \rho(e_1, Fe_1))} \right) \left( \varphi \rho(e_1, Fe_1) + \varphi \rho(e_2, Te_2) \right) \\
\leq \left( \frac{\rho(e_1, e_2)}{\epsilon + \rho(e_1, Fe_1)} \right) \max \left\{ \varphi \rho(e_1, Fe_1), \varphi \rho(e_2, Te_2) \right\}.
\]

Similarly, we can prove that:
\[
\varphi \rho(Te_1, Fe_2) \leq \left( \frac{\rho(e_1, e_2)}{2(\epsilon + \rho(e_1, Fe_1))} \right) \left( \varphi \rho(e_1, Te_1) + \varphi \rho(e_2, Fe_2) \right).
\]

□

**Corollary 5** A complete $(E, q)$ equipped with $\rho$ and let $F, T$ be two self continuous mappings on $E$. Assume the following conditions hold:

(i) For all $e_1, e_2 \in E$ and for a given $\epsilon > 0$ and an ultra distance function $\varphi$ and $k \in [0, 1)$ we have:

\[
\varphi \rho(Fe_1, Te_2) \leq k \varphi \rho(e_1, e_2).
\]

And

\[
\varphi \rho(Te_1, Fe_2) \leq k \varphi \rho(e_1, e_2).
\]

(ii) For all $e_1, e_2 \in E$ we have $\rho(e_1, e_2) \leq L$ for some integer $L$.

Then $F$ and $T$ have a unique common fixed point in $E$. 
Define \( \varphi(\mu_\ast) = \mu_\ast \) and let \( k = (\frac{\rho(e_1, F_1)}{\varepsilon + \rho(e_1, F_1)}) \). Then \( k \in [0, 1) \).

**Proof.** Let \( \varphi(\mu_\ast) = \mu_\ast \) and let \( k = (\frac{\rho(e_1, F_1)}{\varepsilon + \rho(e_1, F_1)}) \). Then \( k \in [0, 1) \).

Now,

\[
\varphi \rho(Fe_1, Te_2) = \rho(Fe_1, Te_2) \\
\leq \frac{\rho(e_1, F(e_2))}{\varepsilon + \rho(e_1, F(e_2))} \rho(e_1, e_2) \\
= \frac{\rho(e_1, e_2)}{\varepsilon + \rho(e_1, e_2)} \rho(e_1, F(e_1)) \\
= \frac{\rho(e_1, e_2)}{\varepsilon + \rho(e_1, e_2)} \varphi \rho(e_1, F(e_1)) \\
\leq \frac{\rho(e_1, e_2)}{\varepsilon + \rho(e_1, e_2)} \max \{ \varphi \rho(e_1, F(e_1)), \varphi \rho(e_2, Te_2) \}.
\]

Similarly, we can prove that:

\[ \varphi \rho(Te_1, Fe_2) \leq k \varphi \rho(e_1, e_2). \]

If we take \( F = T \) in Corollary 5, we get the following result:

**Corollary 6** A complete \((E, q)\) equipped with \( \rho \) and let \( F \) be a self continuous mapping on \( E \). Assume the following conditions hold:

(i) For all \( e_1, e_2 \in E \) and for a given \( \varepsilon > 0 \) and an ultra distance function \( \varphi \) and \( k \in [0, 1) \) we have:

\[ \varphi \rho(Fe_1, Fe_2) \leq k \varphi \rho(e_1, e_2). \]

(ii) For all \( e_1, e_2 \in E \) we have \( \rho(e_1, e_2) \leq L \) for some integer \( L \).

Then \( F \) has a unique common fixed point in \( E \).

**Example 1** Let \( E = 0, 1, \ldots, m \) where \( m \in \mathbb{N} \).

Define \( F, T \) on \( E \) as follows:

\[
F(e_1) = \begin{cases} 
0 & \text{if } e_1 \in \{0, 1\}; \\
1 & \text{if } e_1 \in \{2, 3, \ldots, 5\}; \\
2 & \text{if } e_1 \in \{6, 7, \ldots, m\}. 
\end{cases}
\]

\[
T(e_2) = \begin{cases} 
0 & \text{if } e_2 \in \{0, 1, \ldots, 5\}; \\
1 & \text{if } e_2 \in \{6, 7, \ldots, 10\}; \\
2 & \text{if } e_2 \in \{11, 12, \ldots, m\}. 
\end{cases}
\]

Then \( F \) and \( T \) have a unique fixed point in \( E \).

**Proof.** To show that \( F \) and \( T \) have a unique fixed point in \( E \). Define \( \rho, q : E \times E \to [0, \infty) \) such that

\[
q(e_1, e_2) = \frac{2}{3} e_1 + \frac{1}{3} e_2.
\]

\[
\rho(e_1, e_2) = 2 e_1 + e_2.
\]

Also define \( \varphi(\mu_\ast) : [0, \infty) \to [0, \infty) \) as follows:

\[
\varphi(\mu_\ast) = \begin{cases} 
(1/4) \mu_\ast & \text{if } \mu_\ast \in [0, m]; \\
(1/4)(\mu_\ast^2 + 2) & \text{if } \mu_\ast > m.
\end{cases}
\]

Then

1. \( F \) and \( T \) are continuous functions.

\( \varepsilon \varphi \)-contraction and some fixed point results ... (K. Abodayeh)
2. \( \varphi \) is an ultra distance function.

3. \((E, q)\) is a complete quasi metric space.

4. \( \rho \) is an \( m_\omega \)-distance mapping.

5. The pair \((F, T)\) is \( \epsilon_\varphi \)-contraction with \((\epsilon = 1)\)
i.e., \( \forall e_1, e_2 \in E \) we have

\[
\varphi \rho(F e_1, T e_2) \leq \left( \frac{\varphi \rho(e_1, e_2)}{1 + \varphi \rho(e_1, F e_1)} \right) \max \left\{ \varphi \rho(e_1, F e_1), \varphi \rho(e_2, T e_2) \right\}.
\]

And

\[
\varphi \rho(T e_1, F e_2) \leq \left( \frac{\varphi \rho(e_1, e_2)}{1 + \varphi \rho(e_1, T e_1)} \right) \max \left\{ \varphi \rho(e_1, T e_1), \varphi \rho(e_2, F e_2) \right\}.
\]

Now, it is an easy matter to check out that \( F \) and \( T \) are continuous functions. In addition, it is obviously that \( \varphi \) is an ultra distance function, \( \rho \) is an \( m_\omega \)-distance mapping and \((E, q)\) is a quasi metric space.

To show that \( q \) is complete, let \((e_n)\) be a Cauchy sequence in \( E \). Then for each \( s, t \in \mathbb{N} \) we have

\[
\lim_{s,t \to \infty} q(e_s, e_t) = 0
\]

we conclude that \( e_s = e_t \) for all \( s, t \in \mathbb{N} \) but not for finitely many. Therefore, \((e_n)\) is a convergent sequence in \( E \). Consequently, \((E, q)\) is a complete quasi metric space.

To prove that the pair \((F, T)\) is \( \epsilon_\varphi \)-contraction with \((\epsilon = 1)\), we need to consider the following cases:

Case (1): If \( e_1 \in \{0, 1\} \), then we have the following subcases:

Subcase (1): If \( e_2 \in \{0, 1, \ldots, 5\} \), then

\[
\varphi \rho(F e_1, T e_2) = \varphi \rho(0, 0) = 0.
\]

Subcase (2): If \( e_2 \in \{6, 7, \ldots, 10\} \), then

\[
\varphi \rho(F e_1, T e_2) = \varphi \rho(0, 1) = \varphi(1) = \frac{1}{4}.
\]

\[
\left( \frac{\varphi \rho(e_1, e_2)}{1 + \varphi \rho(e_1, F e_1)} \right) \max \left\{ \varphi \rho(e_1, F e_1), \varphi \rho(e_2, T e_2) \right\} = \left( \frac{\varphi \rho(e_1, e_2)}{1 + \varphi \rho(e_1, 0)} \right) \left[ \frac{1}{2} \varphi \rho(e_2, 1) \right]
\]

\[
= \left( \frac{2 e_1 + 2}{2 e_1 + 1} \right) \left[ \frac{1}{2} (2 e_2 + 1) \right]
\]

\[
\geq \frac{13}{4} \left( \frac{2 e_1 + 6}{2 e_1 + 4} \right)
\]

\[
\geq \left( \frac{13}{4} \right) \left( \frac{1}{4} \right)
\]

\[
\geq \frac{1}{4}.
\]

Subcase (3): If \( e_2 \in \{11, 12, \ldots, m\} \), then we get that

\[
\varphi \rho(F e_1, T e_2) = \varphi \rho(0, 2) = \varphi(2) = \frac{2}{4}.
\]

\[
\left( \frac{\varphi \rho(e_1, e_2)}{1 + \varphi \rho(e_1, F e_1)} \right) \max \left\{ \varphi \rho(e_1, F e_1), \varphi \rho(e_2, T e_2) \right\} = \left( \frac{\varphi \rho(e_1, e_2)}{1 + \varphi \rho(e_1, 0)} \right) \left[ \frac{1}{2} \varphi \rho(e_2, 2) \right]
\]

\[
= \left( \frac{2 e_1 + 2}{2 e_1 + 1} \right) \left[ \frac{1}{2} (2 e_2 + 2) \right]
\]

\[
\geq \left( \frac{2 e_1 + 12}{2 e_1 + 1} \right)
\]

\[
\geq \frac{26}{4}
\]

\[
\geq \frac{1}{4}.
\]
Case (2): If $e_1 \in \{2, 3, \cdots, 5\}$, then we have the following subcases:

Subcase (1): If $e_2 \in \{0, 1, \cdots, 5\}$, then we have

$$
\varphi(p(Fe_1, Te_2) = \varphi(1, 0) = \varphi(2) = \frac{2}{4}.
$$

$$
\left( \frac{\rho(e_1, e_2)}{1 + \rho(e_1, Fe_1)} \right) \max \left\{ \varphi(p(e_1, Fe_1), \varphi(p(e_2, Te_2) \right\} \geq \left( \frac{\rho(e_1, e_2)}{1 + \rho(e_1, 1)} \right) \left[ \frac{1}{4} \rho(e_1, 1) \right]
$$

$$
= \left( \frac{2e_1 + e_2}{2e_1 + 2} \right) \left[ \frac{1}{4} (2e_1 + 1) \right]
$$

$$
\geq \frac{1}{4} \left( \frac{2e_1 + 1}{2} \right)
$$

Subcase (2): If $e_2 \in \{6, 7, \cdots, 10\}$, then we get that

$$
\varphi(p(Fe_1, Te_2) = \varphi(1, 1) = \varphi(3) = \frac{3}{4}.
$$

$$
\left( \frac{\rho(e_1, e_2)}{1 + \rho(e_1, Fe_1)} \right) \max \left\{ \varphi(p(e_1, Fe_1), \varphi(p(e_2, Te_2) \right\} = \left( \frac{\rho(e_1, e_2)}{1 + \rho(e_1, 1)} \right) \left[ \frac{1}{4} \rho(e_2, 1) \right]
$$

$$
= \left( \frac{2e_1 + e_2}{2e_1 + 2} \right) \left[ \frac{1}{4} (2e_1 + 1) \right]
$$

$$
\geq \frac{1}{4} \left( \frac{2e_1 + 1}{12} \right)
$$

Subcase (3): If $e_2 \in \{11, 12, \cdots, m\}$, then we get that

$$
\varphi(p(Fe_1, Te_2) = \varphi(1, 2) = \varphi(4) = 1.
$$

$$
\left( \frac{\rho(e_1, e_2)}{1 + \rho(e_1, Fe_1)} \right) \max \left\{ \varphi(p(e_1, Fe_1), \varphi(p(e_2, Te_2) \right\} = \left( \frac{\rho(e_1, e_2)}{1 + \rho(e_1, 1)} \right) \left[ \frac{1}{4} \rho(e_2, 2) \right]
$$

$$
= \left( \frac{2e_1 + e_2}{2e_1 + 2} \right) \left[ \frac{1}{4} (2e_2 + 2) \right]
$$

$$
\geq 6 \left( \frac{2e_1 + 1}{2e_1 + 2} \right)
$$

$$
\geq \frac{2}{7}
$$

$$
\geq 1.
$$

Case (3): If $e_1 \in \{6, 7, \cdots, m\}$, then we have the following subcases:

Subcase (1): If $e_2 \in \{0, 1, \cdots, 5\}$, then we have

$$
\varphi(p(Fe_1, Te_2) = \varphi(2, 0) = \varphi(4) = 1.
$$

$$
\left( \frac{\rho(e_1, e_2)}{1 + \rho(e_1, Fe_1)} \right) \max \left\{ \varphi(p(e_1, Fe_1), \varphi(p(e_2, Te_2) \right\} = \left( \frac{\rho(e_1, e_2)}{1 + \rho(e_1, 1)} \right) \left[ \frac{1}{4} \rho(e_2, 2) \right]
$$

$$
= \left( \frac{2e_1 + e_2}{2e_1 + 3} \right) \left[ \frac{1}{4} (2e_1 + 3) \right]
$$

$$
\geq \frac{1}{4} \left( \frac{2e_1 + 1}{1} \right)
Subcase (2): If \( e_2 \in \{6, 7, \ldots, 10\} \), then we have

\[
\varphi \rho(F e_1, T e_2) = \varphi \rho(2, 1) = \varphi(5) = \frac{5}{4}.
\]

\[
\left(\frac{\rho(e_1, e_2)}{1 + \rho(e_1, F e_1)}\right) \max \left\{ \varphi \rho(e_1, F e_1), \varphi \rho(e_2, T e_2) \right\} \geq \left(\frac{\rho(e_1, e_2)}{1 + \rho(e_1, F e_1)}\right) \left[\frac{1}{2} \rho(e_1, 2)\right] = \left(\frac{2e_1 + e_2}{2e_1 + 3}\right) \left[\frac{1}{2}(2e_1 + 2)\right] \geq \frac{21}{4}.
\]

Subcase (3): If \( e_2 \in \{11, 12, \ldots, m\} \), then we get that

\[
\varphi \rho(F e_1, T e_2) = \varphi \rho(2, 2) = \varphi(6) = \frac{6}{4}.
\]

\[
\left(\frac{\rho(e_1, e_2)}{1 + \rho(e_1, F e_1)}\right) \max \left\{ \varphi \rho(e_1, F e_1), \varphi \rho(e_2, T e_2) \right\} \geq \left(\frac{\rho(e_1, e_2)}{1 + \rho(e_1, F e_1)}\right) \left[\frac{1}{2} \rho(e_2, 2)\right] = \left(\frac{2e_1 + e_2}{2e_1 + 3}\right) \left[\frac{1}{2}(2e_2 + 2)\right] \geq \frac{6}{4}.
\]

In a similar manner, we can show that:

\[
\varphi \rho(T e_1, F e_2) \leq \left(\frac{\rho(e_1, e_2)}{1 + \rho(e_1, T e_1)}\right) \max \left\{ \varphi \rho(e_1, T e_1), \varphi \rho(e_2, F e_2) \right\}.
\]

Consequently, the pair \((F, T)\) satisfies the conditions of Theorem 3 ensures that \(F\) and \(T\) have a unique common fixed point in \(E\). □

3. APPLICATION

Theorem 7 Let \( m = 2^n \) with \( n \in \mathbb{N} \). Then the function

\[
F(x) = \left(1 - x^m\right)/(\eta - x^m), \quad \text{where } \eta \geq m + 2
\]

has a unique fixed point in \([0, 1]\).

Proof. Let \( E = [0, 1] \). Define \( q : E \times E \to [0, \infty) \) by \( q(e_1, e_2) = |e_1 - e_2| \). Then \((E, q)\) is a complete quasi metric space. Also, define \( \rho : E \times E \to [0, \infty) \) by \( \rho(e_1, e_2) = |e_1 - e_2| \). Then \( \rho \) is an \( m\omega\)-distance mapping. Now, equipped \((E, q)\) with \( \rho \).

Also, define \( \varphi : [0, \infty) \to [0, \infty) \) by

\[
\varphi(\mu) = \begin{cases} 
\mu & \text{if } \mu \in [0, 1]; \\
(1/9)(\mu^2 + 1) & \text{if } \mu > 1.
\end{cases}
\]
Note that $\varphi$ is an ultra distance function. Now,

$$
\varphi \rho(Fe_1, Fe_2) = \left| \frac{1 - e_1^m}{\eta - e_1^m} - \frac{1 - e_2^m}{\eta - e_2^m} \right|
= \left| \frac{(1 - e_1^m)(\eta - e_2^m) - (1 - e_2^m)(\eta - e_1^m)}{(\eta - e_1^m)(\eta - e_2^m)} \right|
= \left| \frac{(\eta - 1)(\eta - e_1^m)(\eta - e_2^m)}{(\eta - e_1^m)(\eta - e_2^m)} \right| \left| e_1^m - e_2^m \right|
= \left| e_1^m - e_2^m \right|
\leq \frac{(\eta - 1)(2^n)}{(\eta - 1)^2} \left| e_1 - e_2 \right|
= \frac{(\eta - 1)(m)}{(\eta - 1)^2} \left| e_1 - e_2 \right|
= \frac{(\eta - 1)(m)}{(\eta - 1)^2} \varphi \rho(e_1, e_2).
$$

By taking $k = \frac{(\eta - 1)(m)}{(\eta - 1)^2}$ then $k < 1$ and noting that $F$ is continuous, we deduce that $F$ satisfies all conditions of Corollary 6. Therefore, $F$ has a unique fixed point in $E$. \(\square\)

**Example 2** The function 

$$
F(x) = \left[ \frac{1 - x^{128}}{130 - x^{128}} \right]
$$

has a unique fixed point in \([0, 1]\).

**Proof.** By applying Theorem 7 with $m = 128$ and $\eta = 130$. \(\square\)

4. **CONCLUSION**

Based on the definition of modified $\omega$-distance mappings, the notion of the $\epsilon_\varphi$-contraction was introduced. By employ this new definition, we proved some fixed point result. An example was introduced to show the validity and reliability of our new results.

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